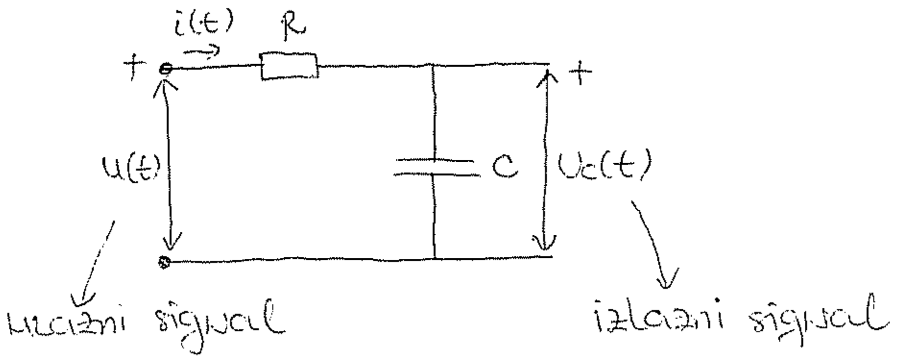
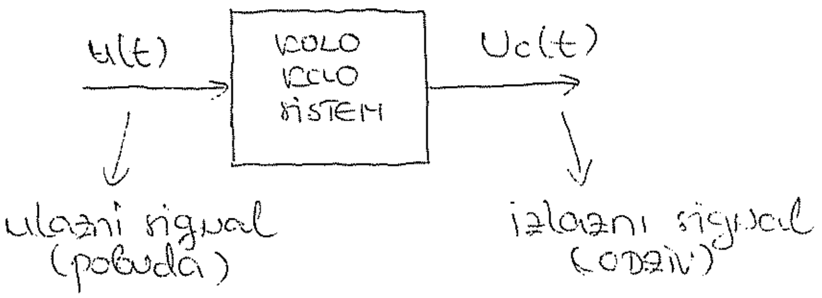
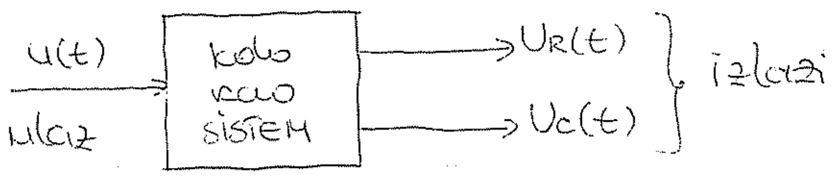
PREDAVANJE

SIGNAL – skup nekih informacija i podataka. Mi posmatramo signale koje su funkcije vreme.

SISTEM – objekat koji obrađuje skup nekih signala koje nazivamo ulazni signali ili pobuda i proizvodi skup nekih drugih signala koji se nazivaju izlazni signali ili odziv.

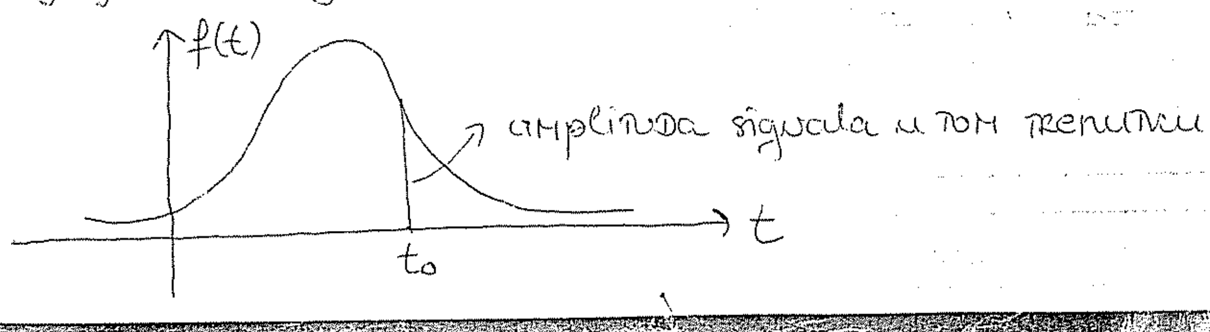




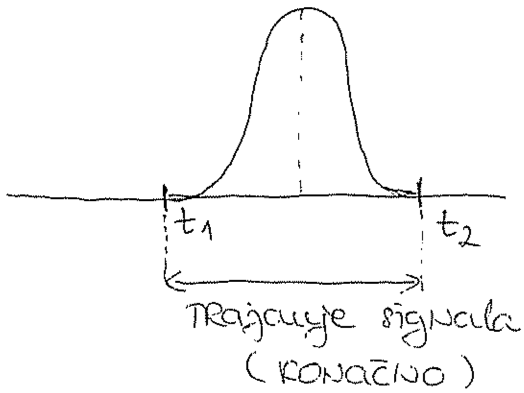


NEKE VAŽNE VELIČINE VEZANE ZA SIGNAL

1° **VELIČINA ILI AMPLITUDA SIGNALA** – broj koji pokazuje veličinu (tj jačinu) signala u datom trenutku.

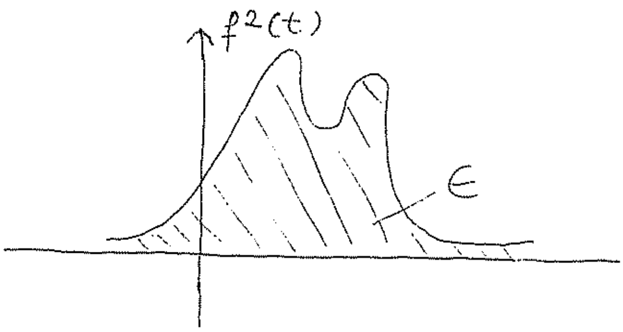


2° TRAJANJE SIGNALA – ako signal postoji kada vrijeme teži $\pm\infty$ onda je trajanje signala beskonačan.
Postoje signali koji imaju konačno trajanje.



3° ENERGIJA SIGNALA

$$f(t) \rightarrow f^2(t) \geq 0$$



Energija signala je površina ispod ove krive na slici a to je određeni integral.

$$\boxed{E = \int_{-\infty}^{+\infty} f^2(t)\,dt}$$ Definicija energije

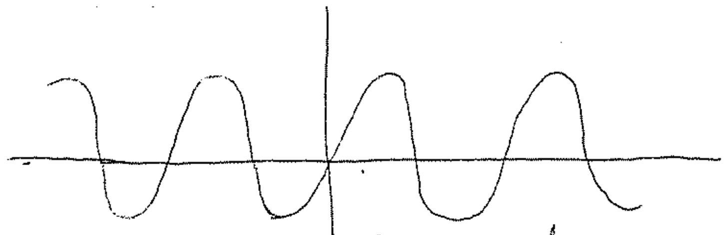
Ako je signal konačnog trajanja energija je uvijek konačna.

$t \rightarrow \pm\infty \quad f^2(t) \rightarrow 0$ (brzo teži 0-li i tada je energija konačna)

4° SNAGA SIGNALA

$$\boxed{P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} f^2(t)\,dt}$$

Za signal trajanja snaga je uvijek konačna.
Postoje beskonačni signali za koje je snaga konačna.
Jedan primjer za to je periodični signal.



Periodični signal

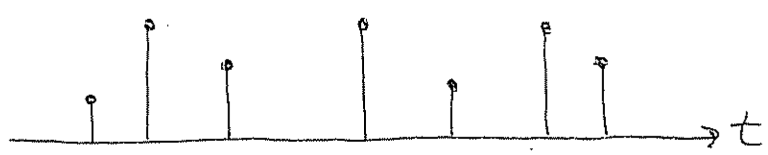
Klasifikacija signala

Postoji više klasifikacija:

1° Vremenski kontinualne i vremenski diskretne

Vremenski kontinualan je signal definisan za svaki vremenski trenutak.

Vremenski diskretni je signal koji postoji samo u određenim vremenskim trenucima.

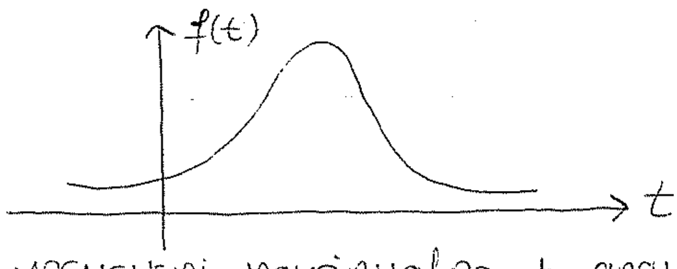


Vremenski diskretni signal

2° Analogni i digitalni signali

Analogni signal je signal čija amplituda može uzimati sve vrijednosti iz nekog opsega.

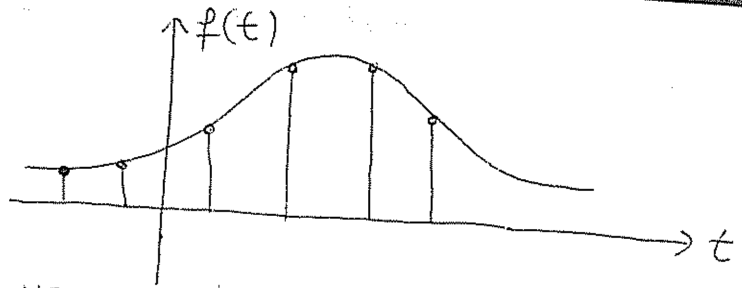
Digitalni signal je onaj signal čija amplituda može da uzima samo određeni broj nekih diskretnih vrijednosti.



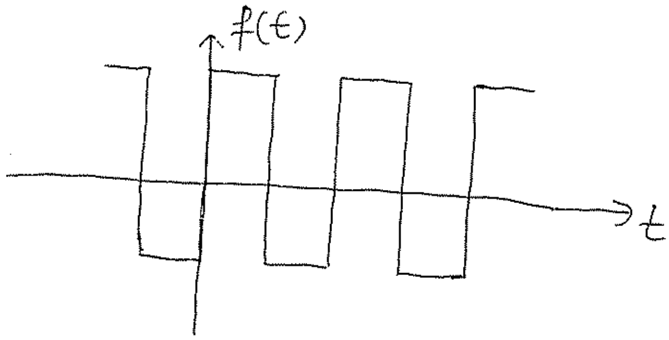
Vremenski kontinualna + analogni signal

2

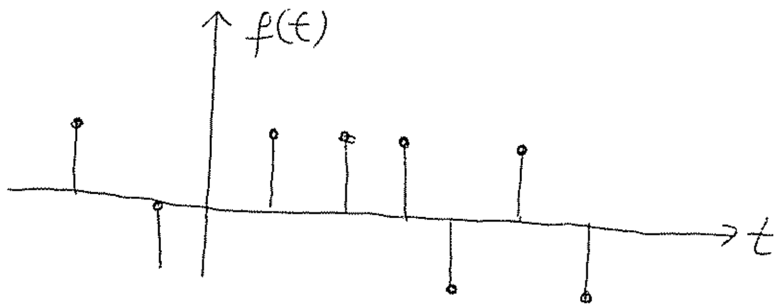
T



VREMENSKI DISKRETNI + analogni signal



VREMENSKI KONTINUALNI + DIGITALNI SIGNAL
i zauzima samo dvije vrijednosti



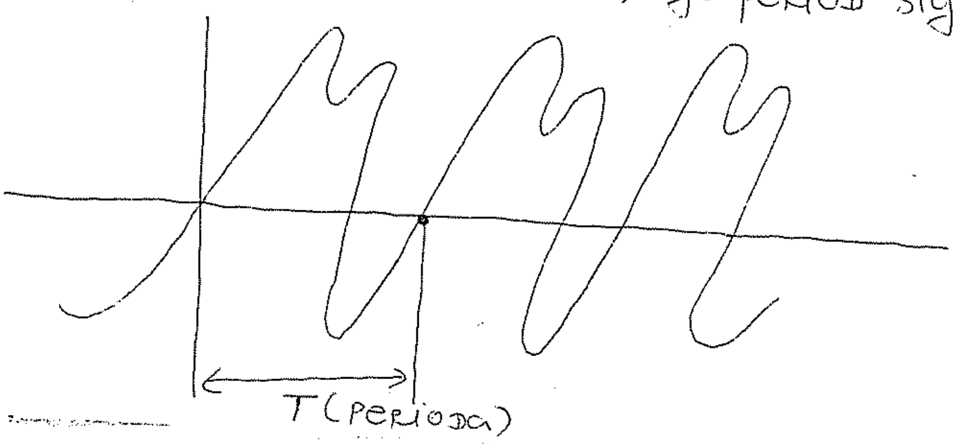
VREMENSKI DISKRETNI + DIGITALNI SIGNAL
i zauzima dvije vrijednosti

3° PERIODIČNI i NEPERIODIČNI SIGNALI

Signal je periodičan ako zadovoljava relaciju:

$$f(t+T_0)=f(t) \quad \forall t$$

najmanji broj T_0 ($\min T_0 = T$) je period signala



4° ENERGETSKI SIGNALI I SIGNALI SNAGE

Energetski signal je signal ako mu je energija konačna.
Signal snage je signal ako mu je snaga konačna.

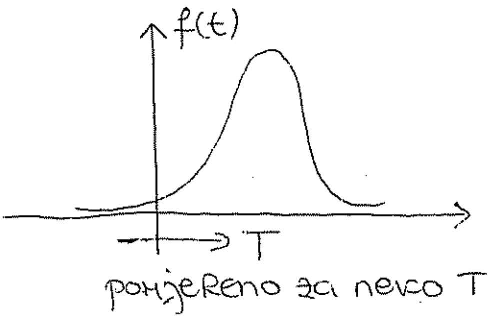
5° DETERMINISTIČKI I SLUČAJNI (STOHASTIČKI) SIGNALI

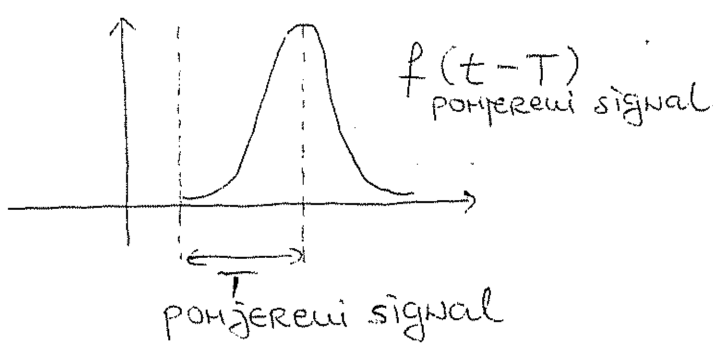
Deterministički signal je signal ako se može opisati egzaktno u svakom trenutku (analitički preko formule ili grafički).

Ukoliko vrijednost signala nije poznata tačno u nekim vremenskim trenucima onda taj signal uzimamo slučajnim ili stohastički.

NEKE PROSTE OPERACIJE SA SIGNALIMA

1° VREMENSKO POMIJERANJE SIGNALA



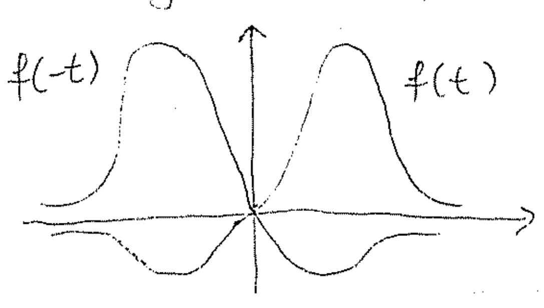


$T>0$ pomijeranje se vrši udesno (ako je T pozitivno)
$T<0$ pomijeranje se vrši ulijevo (ako je T negativno)

2° VREMENSKA INVERZIJA SIGNALA

– se dobija (invertovan signal) kada se u signalu t zamjeni sa $-t$

$f(t)$ – signal $\quad$ $f(-t)$ – vremenski invertovan signal

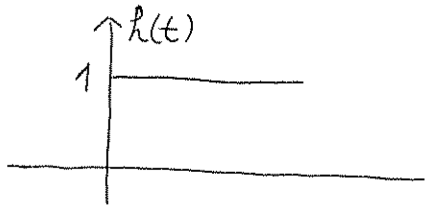


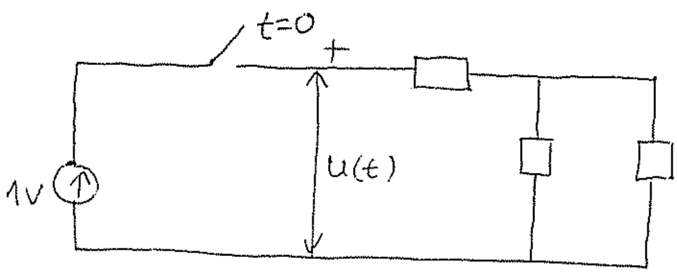
Primjeri nekih često korišćenih signala

1° Jedinična (Hevisajdova) funkcija

- označava se sa h

$$h(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$





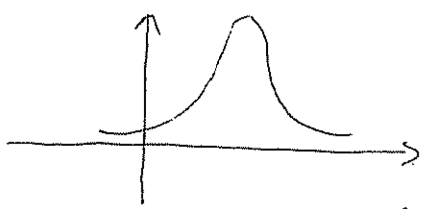
kada se prekidač zatvori
$u(t) = 1V$

$$U(t) = h(t)$$

$$\mathcal{E} = 10V \qquad U(t) = 10\,h(t)$$

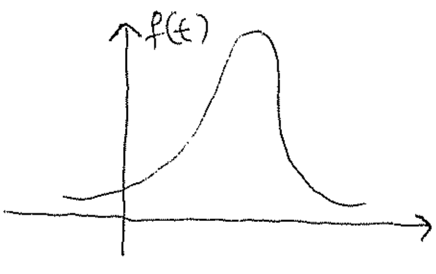
Jedinična funkcija je primjer tz. kauzalnog signala. To je signal koji je jednak nuli za $T<0$, pojavljuje se tek za $T=0$.

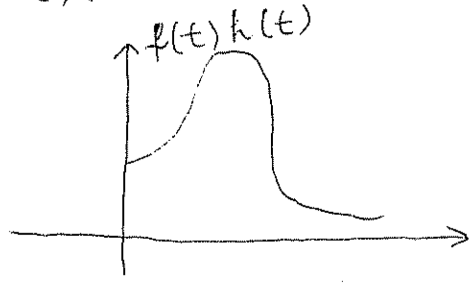
Ako signal postoji i u trenucima $T<0$ naziva se nekauzalni signal.

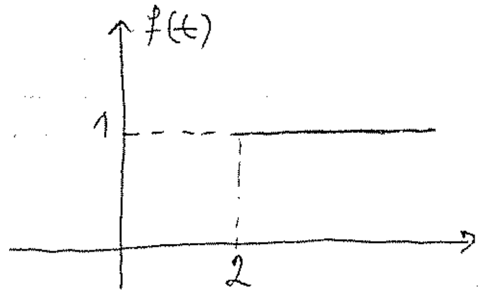


nekauzalni signal

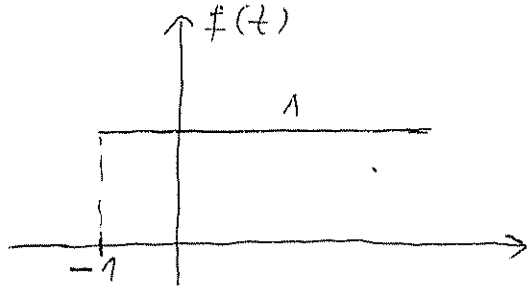
Svaki nekauzalni signal postaje kauzalni ako se pomnoži sa jediničnom funkcijom $h(t)$.



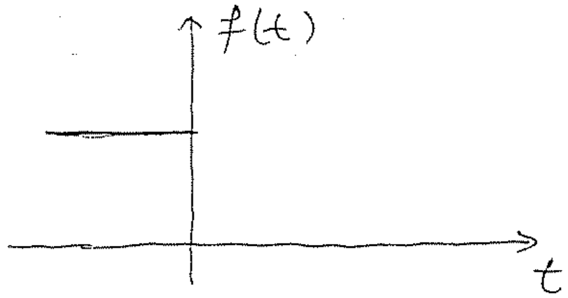




$f(t)=h(t-2)$



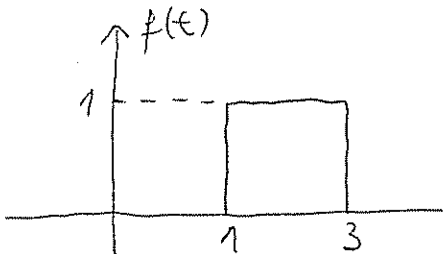
$f(t)=h(t+1)$



→ invertovana jedinična funkcija

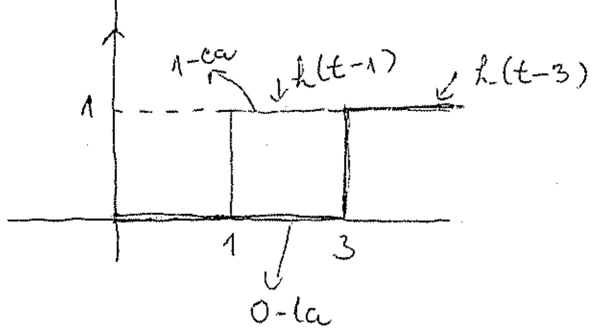
$f(t)=h(-t)$

Jedinična funkcija se često koristi za opisivanje nekih drugih signala.



$h(t)=h(t-1)-h(t-3)$

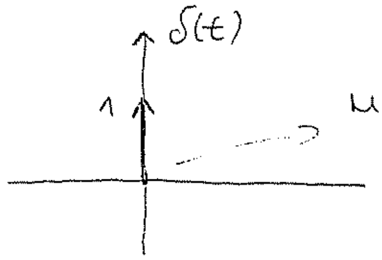
razlika ove dvije funkcije



2° JEDINIČNA IMPULSNA (DIRAHOVA) FUNKCIJA

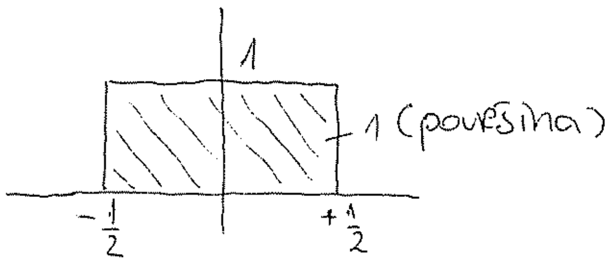
- označava se sa δ

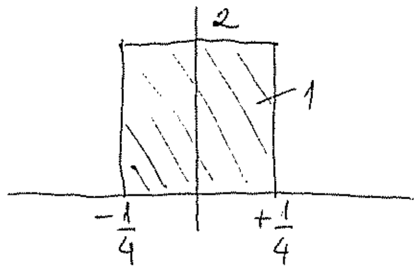
$$\delta(t)=\begin{cases}0 & t\neq 0\\ \infty & t=0\end{cases} + \int_{-\infty}^{+\infty}\delta(t)\,dt=1$$

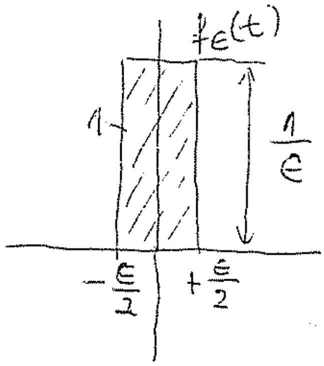


u tu tačku ima beskonačnu vrijednost i označavamo sa 1-com.

Jedinična impulsna funkcija nije obična funkcija nego dijalizovana funkcija. Ova funkcija se može aproksimirati običnim funkcijama koje zadovoljavaju drugi uslov (tj. $\int_{-\infty}^{+\infty} \delta(t)\,dt = 1$), a djelimično prvi uslov $\left(\begin{matrix} 0 & t \neq 0 \\ \infty & t = 0 \end{matrix}\right)$.





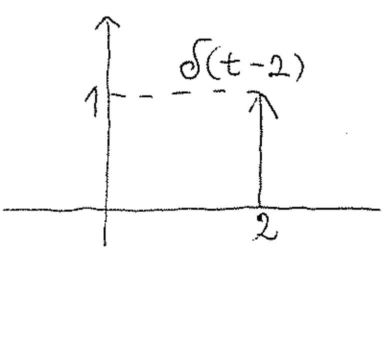


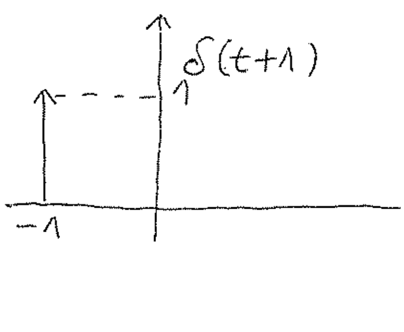
$\epsilon \to 0$

$$\boxed{\delta(t) = \lim_{\epsilon \to 0} f_\epsilon(t)}$$

ϵ vrlo malo

$f_\epsilon(t) \approx \delta(t)$





$$\boxed{\int_{-\infty}^{+\infty} f(t)\,\delta(t)\,dt = f(0)\int_{-\infty}^{+\infty}\delta(t)\,dt = f(0)}$$ → koristi se u zadacima

$$\boxed{\int_{-\infty}^{+\infty} f(t)\,\delta(t-t_0)\,dt = f(t_0)}$$

3° EKSPONENCIJALNA FUNKCIJA (SIGNAL)

$f(t) = e^{st}$

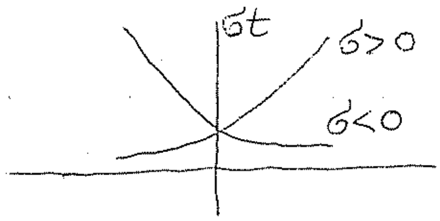
s je kompleksan broj

$s = \sigma + j\omega$

↙ realni broj ↘ imaginarni broj

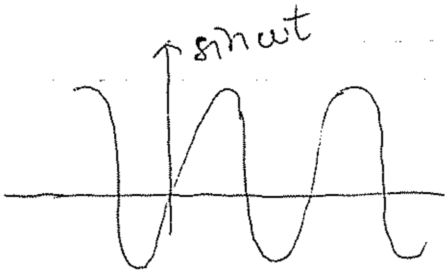
ⓐ $s=0$ dobijamo $f(t) = 1 = const$

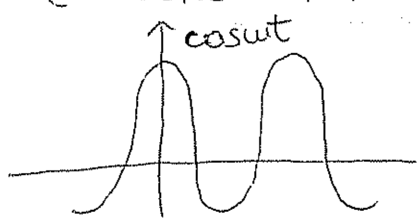
ⓑ $\sigma \neq 0$ $\omega \neq 0$ dobijamo $f(t) = e^{\sigma t}$



ⓒ $\sigma = 0$ $\omega \neq 0$ dobijamo $f(t) = e^{j\omega t} = \cos\omega t + j\sin\omega t$

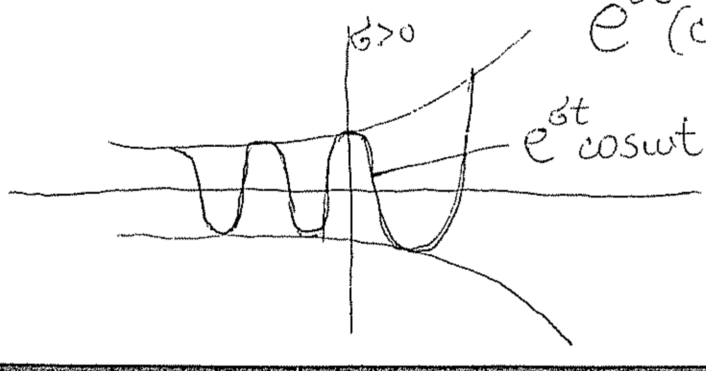
(Ojlerova formula)





ⓓ $\sigma \neq 0$ $\omega \neq 0$ dobijamo $f(t) = e^{(\sigma + j\omega)t} = e^{\sigma t}\cdot e^{j\omega t} =$

$e^{\sigma t}(\cos\omega t + j\sin\omega t)$ $\begin{cases} e^{\sigma t}\cos\omega t \\ e^{\sigma t}\sin\omega t \end{cases}$



ZADACI ZA VJEŽBU:

Nacrtati sledeće signale:

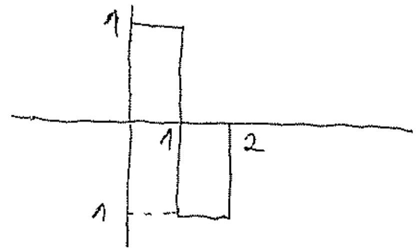
1° $f(t) = h(t) \sin \omega t$

2° $f(t) = e^{-t} h(t)$

3° $f(t) = h(-t-1) - h(t+1)$

4° $f(t) = 2h(-t+2)$

5°



Napisati analitički oblik ovog signala $f(t) = ?$

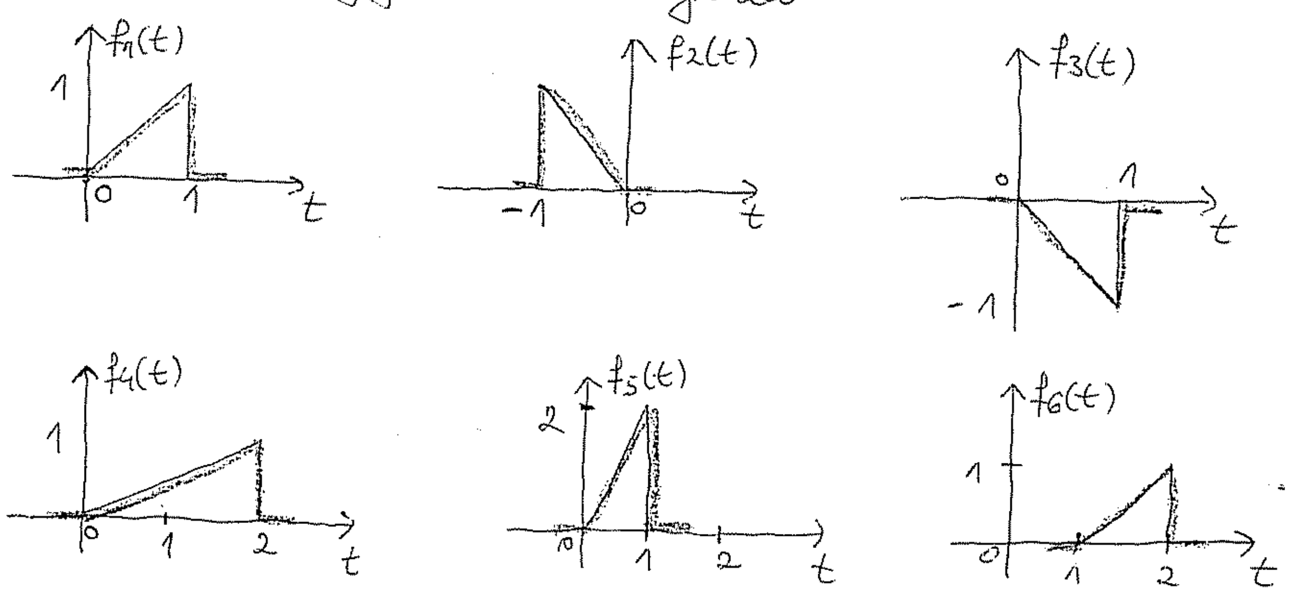
Uputstvo: $f(t) = Ah(t) + Bh(t-1) + Ch(t-2)$

$A = ?$ $B = ?$ $C = ?$

Rješenje: $f(t) = h(t) - 2h(t-1) + h(t-2)$

VJEŽBE

① Odrediti energiju sledećih signala:



$$f_1(t) = \begin{cases} 0, & t < 0 \text{ i } t > 1 \\ t, & 0 \le t \le 1 \end{cases}$$

$$E_{f_1} = \int_0^1 t^2 dt = \frac{t^3}{3} \Big|_0^1 = \frac{1}{3}$$

$$f_2(t) = \begin{cases} 0 , & t<-1 \text{ i } t>0 \\ -t , & -1 \leq t \leq 0 \end{cases}$$

$$E_{f_2} = \int_{-1}^{0} (-t)^2 dt = \frac{t^3}{3} \Big|_{-1}^{0} = \frac{1}{3}$$

$$f_3(t) = \begin{cases} 0 , & t<0 \text{ i } t>1 \\ -t , & 0 \leq t \leq 1 \end{cases}$$

$$E_{f_3} = \int_{0}^{1} (-t)^2 dt = \frac{t^3}{3} \Big|_{0}^{1} = \frac{1}{3}$$

$$f_4(t) = \begin{cases} 0, & t<0 \text{ i } t>2 \\ t/2 , & 0 \leq t \leq 2 \end{cases}$$

$$E_{f_4} = \int_{0}^{2} (t/2)^2 dt = \frac{1}{4} \int_{0}^{2} t^2 dt = \frac{1}{4} \cdot \frac{t^3}{3} \Big|_{0}^{2} = \frac{2}{3}$$

$$f_5(t) = \begin{cases} 0 , & t<0 \text{ i } t>1 \\ 2t , & 0 \leq t \leq 1 \end{cases}$$

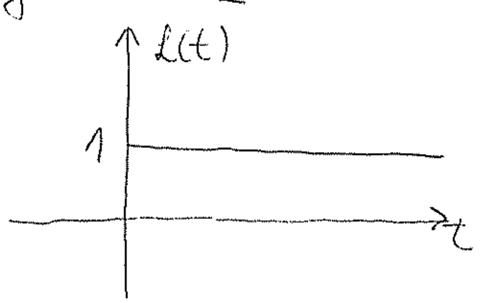
$$E_{f_5} = \int_{0}^{1} (2t)^2 dt = 4 \int_{0}^{1} t^2 dt = 4 \, \frac{t^3}{3} \Big|_{0}^{1} = \frac{4}{3}$$

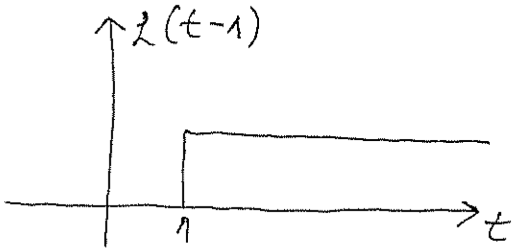
$$f_6(t) = \begin{cases} 0 , & t<1 \text{ i } t>2 \\ t-1 , & 1 \leq t \leq 2 \end{cases}$$

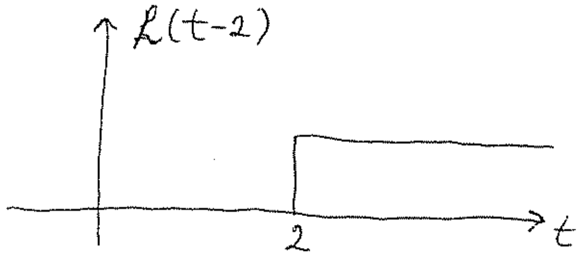
$$E_{f_6} = \int_{1}^{2} (t-1)^2 dt = \int_{1}^{2} (t^2 - 2t + 1) dt = \int_{1}^{2} t^2 dt - 2 \int_{1}^{2} t dt + \int_{1}^{2} dt = \frac{1}{3}$$

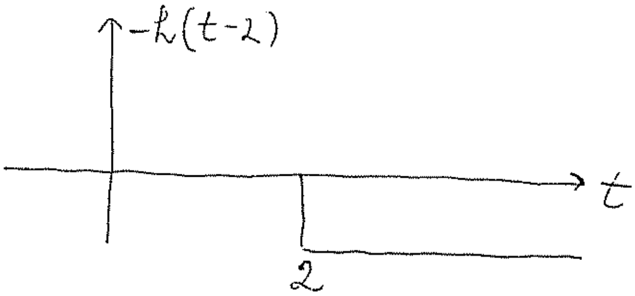
② Nacrtati signal

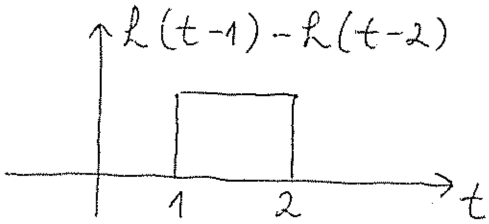
$$g(t) = t^2 \left[\mathcal{L}(t-1) - \mathcal{L}(t-2) \right]$$

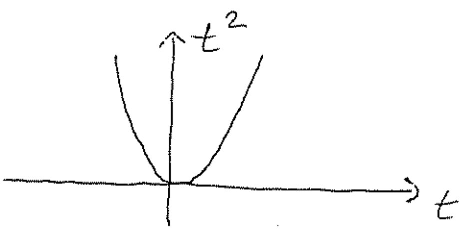


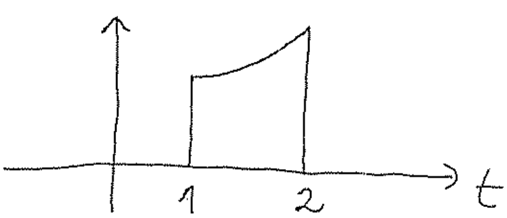








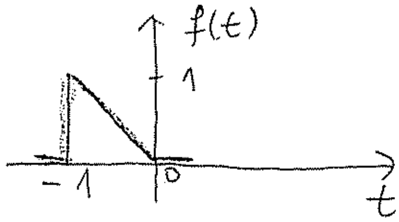


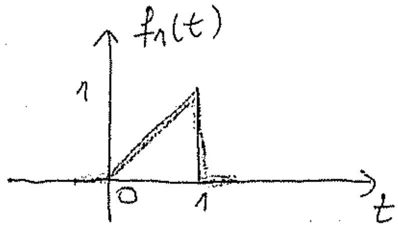


ZA VJEŽBU

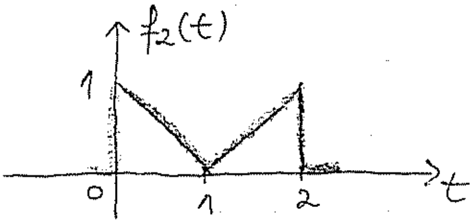
$$g_1(t) = \frac{t}{3}\left[2h(t-1) - 2h(t-3)\right]$$

③ Izraziti signale $f_1(t), f_2(t), f_3(t), f_4(t), f_5(t)$ preko signala $f(t)$

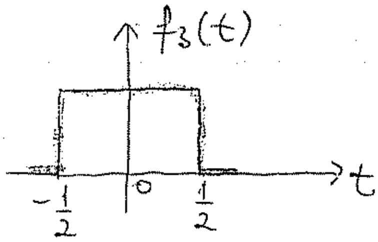




$f_1(t) = f(-t)$ inverzno preslikavanje

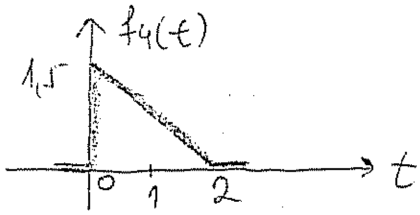


$$f_2(t) = f(t-1) + f(-t-1)$$



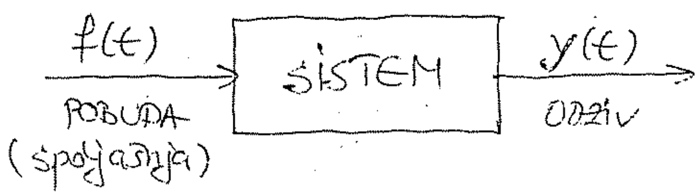
$$f_3(t) = f\left(t-\frac{1}{2}\right) + f\left(-t+\frac{1}{2}\right)$$

ZA VJEŽBU

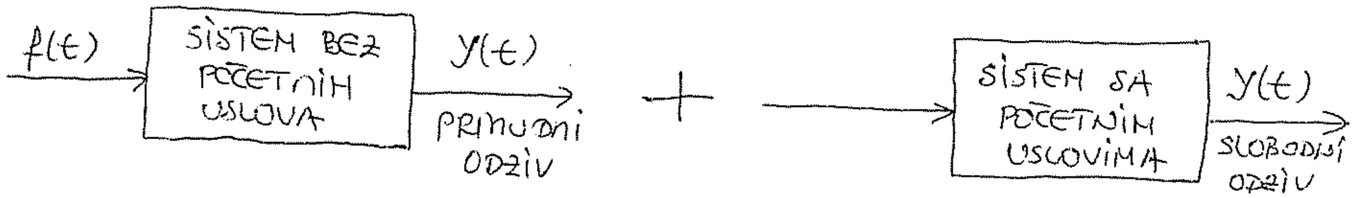


PREDAVANJE

SISTEMI



Unutrašnja pobuda je rezultat tz. početnih uslova. Sam sistem može da ima svoju sopstvenu pobudu (unutrašnja pobuda). Ona je posljedica početne energije odnosno početnih uslova koji postoje u sistemu prije uključenja. Zbog toga analizu sistema obično vršimo u dva koraka.



KLASIFIKACIJA SISTEMA

1° LINEARNI I NELINEARNI SISTEMI
↓
POSMATRAĆEMO SAMO LINEARNE

2° VREMENSKI INVARIJANTNI ILI NEPROMJENLJIVI SISTEMI I VREMENSKI PROMJENLJIVI SISTEMI

parametri sistema ne zavise od vremena.

3° KAUZALNI I NEKAUZALNI SISTEMI

Kauzalni znači ne postoji ODZIV prije vremena $t=0$.

4° KONTINUALNI I DISKRETNI SISTEMI

5° ANALOGNI I DIGITALNI SISTEMI

LINEARNI SISTEMI

$f_1(t) \longrightarrow y_1(t)$

↓
je uvijek pobuda

$f_2(t) \longrightarrow y_2(t)$

$$\left.\begin{array}{l} f_1(t)+f_2(t) \longrightarrow y_1(t)+y_2(t) \\ k f_1(t) \longrightarrow k y_1(t) \end{array}\right\}$$

ako je uslov ispunjen sistem je linearan, ako ne važi sistem je nelinearan

↓
konstanta

Opisani sistem znači veza između pobude kao ulaza i odziva kao izlaza, i na osnovu te veze naći odziv kada je pobuda poznata. Sistem se obično opisuje tz. DIFERENCIJALNOM JEDNAČINOM.

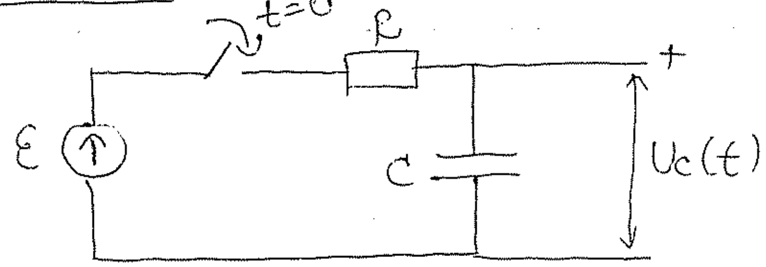
Diferencijalna jednačina je jednačina u kojoj se pored nepoznatog odziva tj. $y(t)$ srijeću i izvodi te funkcije. Sistem je prvog reda ako se opisuje diferencijalnom jednačinom prvog reda $y'(t)$, drugog reda ako imamo prvi i drugi izvod $y'(t)$, $y''(t)$.

Da bi našli odziv potrebno je riješiti neku diferencijalnu jednačinu. Riješiti diferencijalnu jednačinu znači naći funkciju koja tu jednačinu indentički zadovoljava, i koja još zadovoljava i date početne uslove.

Kod linearnih sistema ukupan odziv je suma prinudnog i slobodnog odziva.

$$\underbrace{y(t)}_{\text{ukupni odziv}} = y_{\substack{\text{prinudni}\\ \text{odziv}}}(t) + y_{\substack{\text{slobodni}\\ \text{odziv}}}(t)$$

PRIMJER



ODZIV može biti: struja kroz kolo ...

$\mathcal{E} \equiv f(t)$ pobuda

$U_c(t) \equiv y(t)$ odziv (prinudan)

$t<0$

nema struje $i(t)=0$

nema napona na kondenzatoru $U_c(t)=0$

to je kauzalan sistem, kontinualan sistem, vremenski invarijantni

$t>0$

$U_c(t)=?$ $\quad \mathcal{E}$ poznato

$$\mathcal{E}=Ri(t)+U_c(t)$$

$$\boxed{U_c(t)=\frac{q(t)}{C}}$$

$$Ri(t)+\frac{q(t)}{C}=\mathcal{E}$$

$$i(t)=\frac{dq(t)}{dt}$$

$$\boxed{R\frac{dq(t)}{dt}+\frac{q(t)}{C}=\mathcal{E}} \quad (1)$$

diferencijalna jednačina I reda

za $t=0$

$$\boxed{q(0)=0} \quad (2)$$

početni uslov

Rješenje jednačine (1) je:

$$q(t)=C\mathcal{E}-k\cdot e^{-\frac{t}{RC}}$$

nepoznata konstanta

$$\frac{dq}{dt}=-k\cdot e^{-\frac{t}{RC}}\left(-\frac{1}{RC}\right)$$ → ovo zamijenimo u jednačinu (1)

$$R\cdot\left(\frac{k}{RC}\cdot e^{-\frac{t}{RC}}\right)+\frac{1}{C}\left(C\mathcal{E}-k\cdot e^{-\frac{t}{RC}}\right)=$$

$$\frac{k}{C}\cdot e^{-\frac{t}{RC}}+\mathcal{E}-\frac{k}{C}\cdot e^{-\frac{t}{RC}}=\mathcal{E}$$

$$\mathcal{E}=\mathcal{E} \quad \text{rješenje}$$

za $t=0$ treba i da $q(0)=0=C\mathcal{E}-k \Rightarrow \boxed{k=C\mathcal{E}}$

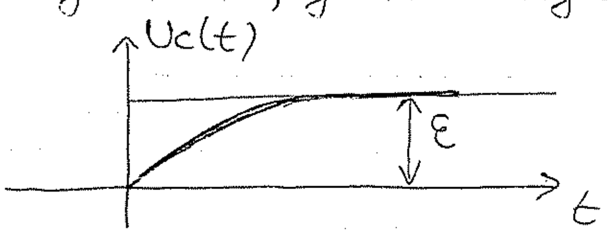
$$q(t)=C\mathcal{E}-C\mathcal{E}\cdot e^{-t/RC}=$$

$$\boxed{C\mathcal{E}\left(1-e^{-t/RC}\right)}$$

rješenje q za svaki trenutak

$$\boxed{U_C(t)=\frac{q(t)}{C}=\mathcal{E}\left(1-e^{-t/RC}\right)} \quad \text{za } t>0$$

To je odziv, grafički izgled odziva:



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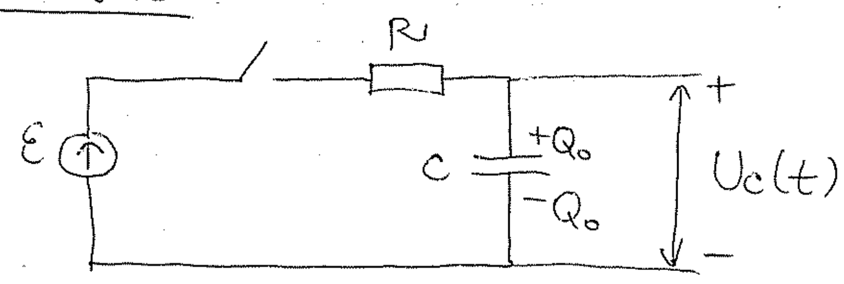


$t<0 \quad U_C(t)$

Prinudni odziv znači nema početni uslov.

$$y_{prinudni}\equiv\boxed{U_C(t)=\mathcal{E}\left(1-e^{-\frac{t}{RC}}\right)} \quad \text{prinudni odziv}$$

Slobodni odziv naći ćemo ako je slobodni odziv $=0$

$$\mathcal{E} \equiv 0$$

$$R\frac{dq}{dt} + \frac{1}{C}q = 0$$

$$q(t) = k \cdot e^{-\frac{t}{RC}}$$

$$\boxed{U_C(t) = \frac{k}{C} \cdot e^{-\frac{t}{RC}}}$$ slobodni odziv

$$U_C(t) = \mathcal{E}\left(1 - e^{-\frac{t}{RC}}\right) + \frac{k}{C} \cdot e^{-\frac{t}{RC}}$$

$t=0$ $\quad q(t) = Q_0$

$$q(0) = Q_0$$

$$U_C(t) = \frac{q(t)}{C}$$

$$\boxed{U_C(0) = \frac{q(0)}{C} = \frac{Q_0}{C}}$$ za $t=0$

$$U_C(0) = \frac{k}{C} = \frac{Q_0}{C}$$

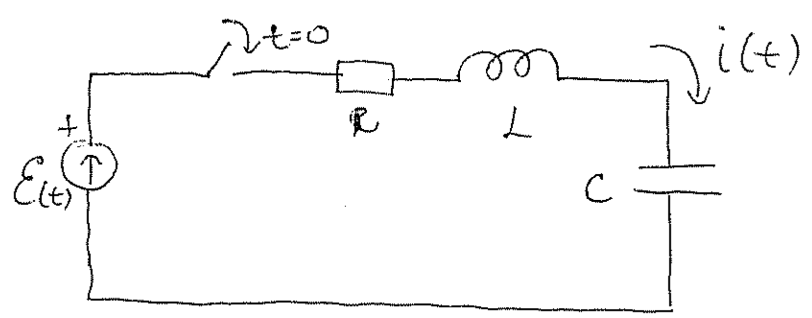
$$k = Q_0$$

$$\boxed{U_C(t) = \mathcal{E}\left(1 - e^{-\frac{t}{RC}}\right) + \frac{Q_0}{C} \cdot e^{-\frac{t}{RC}}}$$

ukupni odziv na kondenzatoru

PRIMJER

SISTEM II REDA



za $t<0 \quad i(t)=0$

za $t>0 \quad i(t)=?$ $\quad \mathcal{E}(t)$ poznato

$$\boxed{\mathcal{E}(t) = Ri(t) + L\frac{di(t)}{dt} + U_c(t)}$$

DIFERENCIJALNA JEDNAČINA II REDA

$$Ri(t) + L\frac{di(t)}{dt} + \frac{q(t)}{C} \quad \Big| \frac{d}{dt}$$

$$\frac{d\mathcal{E}(t)}{dt} = R\underbrace{\frac{di(t)}{dt}}_{\text{I IZVOD}} + L\underbrace{\frac{d^2i(t)}{dt^2}}_{\text{II IZVOD}} + \frac{1}{C}i(t)$$

$$\frac{dq}{dt} = i$$

$$L\frac{d^2i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = \frac{d\mathcal{E}(t)}{dt} \quad /:L$$

$$\boxed{i''(t) + \frac{R}{L}i'(t) + \frac{1}{LC}i(t) = \frac{1}{L}\mathcal{E}'(t)} + \text{POČETNI USLOVI} \quad (1)$$

DIFERENCIJALNA JEDNAČINA II REDA

Rješenje ove jednačine je $i(t)$ tj. prinudni odziv

$i(0)=0$ I početni uslov

$q(0)=0$ II početni uslov

$q'(0)=0$ -||-

$$\boxed{\frac{d}{dt} = D}$$ → 'D' nije broj nego operator koji označava izvod

oznaka I izvoda

$$Dy = \frac{dy}{dt} = y'$$

$$\boxed{\frac{d^2}{dt} = D^2} \rightarrow \text{II izvod} \qquad \boxed{\frac{d^3}{dt} = D^3} \rightarrow \text{III izvod}$$

jednačina (1) postaje:

$$D^2 i + \frac{R}{L} Di + \frac{1}{LC} i = \frac{1}{L} D\mathcal{E}$$

$$\boxed{\left(D^2 + \frac{R}{L} D + \frac{1}{LC}\right) i = \frac{1}{L} D\mathcal{E}}$$

ovo je isto što i jednačina (1)

U opštem slučaju sistem će biti opisan diferencijalnom jednačinom koja može biti n-tog reda.

$$\frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) =$$

$$= b_m \frac{d^m f(t)}{dt^m} + b_{m-1} \frac{d^{m-1} f(t)}{dt^{m-1}} + \dots + b_1 \frac{df(t)}{dt} + b_0 f(t)$$

na desnoj strani treba da bude pobuda

$m \leq n$

$$D^n y + a_{n-1} D^{n-1} y + \dots + a_1 Dy + a_0 y = b_m D^m f + \dots + b_1 Df + b_0 f$$

$$\underbrace{(D^n + a_{n-1} D^{n-1} + \dots + a_1 D + a_0)}_{\text{polinom po } D \quad Q(D)} y = \underbrace{(b_m D^m + \dots + b_1 D + b_0)}_{\text{polinom po } D \quad P(D)} f$$

$$\boxed{Q(D)\underbrace{y(t)}_{\text{odziv koji se traži}} = P(D)\underbrace{f(t)}_{\text{pobuda}}} \qquad (1)$$

Da bi našli odziv iz ove jednačine prvo nalazimo prinudni odziv tj. rješenje jednačine (1) bez početnih uslova.

$$Q(D)\underbrace{y(t)}_{\text{prinudni } (t)} = P(D) f(t)$$

$$\left.\begin{array}{l} y(0)=0 \\ y'(0)=0 \\ y''(0)=0 \\ \vdots \\ y^{(n-1)}(0)=0 \end{array}\right\} \; y_{\text{prinudni}}(t)$$

$Q(D)y(t)=0$ + početni uslov koji nisu svi jednaki 0

$\downarrow y_{\text{slobodno}}(t)$

$$y(t)=y_{\text{prinudni}}(t)+y_{\text{slobodni}}(t)$$

Nalaženje slobodnog odziva

$Q(D)y(t)=0$ (sa početnim uslovima)

$y(t)=Ce^{\lambda t}$

$$(D^n+a_{n-1}D^{n-1}+\dots+a_1D+a_0)y(t)=0$$

$$Dy(t)=\frac{dy(t)}{dt}=C\cdot e^{\lambda t}\cdot\lambda$$

$$D^2y(t)=\frac{d^2yt}{dt^2}=C\lambda^2\cdot e^{\lambda t}$$

$\vdots$

$$D^ny(t)=C\cdot\lambda^n\cdot e^{\lambda t}$$

$$C\lambda^n e^{\lambda t}+C\lambda^{n-1}a_{n-1}e^{\lambda t}+\dots+Ca_1\lambda e^{\lambda t}+Ca_0e^{\lambda t}=0$$

$$C\cdot e^{\lambda t}(\lambda^n+a_{n-1}\lambda^{n-1}+\dots+a_1\lambda+a_0)=0 \;/ : Ce^{\lambda t}$$

$$\boxed{\lambda^n+a_{n-1}\lambda^{n-1}+\dots+a_1\lambda+a_0=0}$$

Pretpostavimo da ova jednačina ima n različitih rješenja $\lambda_1,\lambda_2,\dots\lambda_n$ i realna rješenja.

$\lambda_1 \rightarrow C_1 e^{\lambda_1 t}$

$\lambda_2 \rightarrow C_2 e^{\lambda_2 t}$

$\lambda^n \rightarrow C_n e^{\lambda n t}$

Rješenje odziva (jednačine) je:

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \ldots + C_n e^{\lambda n t}$$

slobodni odziv

Konstante $C_1, C_2 \ldots C_n$ ćemo naći iz zadanih početnih uslova, treba da ih bude n.

$$\left.\begin{array}{l} y(0) \\ y'(0) \\ y''(0) \\ \vdots \\ y^{(n-1)}(0) \end{array}\right\} \text{početni uslovi}$$

PRIMJER

Riješiti jednačinu II reda.

$y''(t) + 3y'(t) + 2y(t) = 0$

$y(0) = 0$

$y'(0) = 1$

$(D^2 + 3D + 2) y(t) = 0$

$\lambda^2 + 3\lambda + 2 = 0$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{4ac}}{2} = \frac{-3 \pm \sqrt{9+8}}{2} = \frac{-3 \pm 1}{2}$$

$\lambda_1 = -2 \qquad \lambda_2 = -1$

$$\boxed{y(t) = C_1 e^{-2t} + C_2 e^{-t}} \quad (1')$$

$t=0$

$y(0) = \boxed{0 = C_1 + C_2}$ (1) jednačina

$y'(t) = (-2)C_1 e^{-2t} + C_2 e^{-t}(-1)$

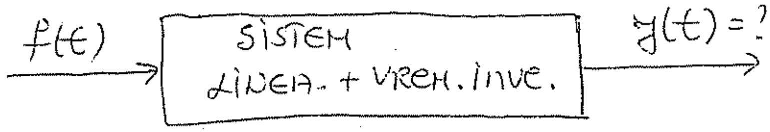
$t=0$

$y'(0) = \boxed{1 = -2C_1 - C_2}$ (2) jednačina

$C_1 = -1$

$C_2 = 1$

$\boxed{y(t) = -e^{-2t} + e^{-t}}$ UVRSTIMO U (1')



$\boxed{Q(D)y(t) = P(D)f(t)}$ (1)

+ početni uslovi

LINEARNA DIFERENCIJALNA JEDNAČINA

$Q(D) = D^n + a_{n-1}D^{n-1} + \ldots + a_1 D + a_0$

$P(D) = b_m D^n + b_{m-1} D^{n-1} + \ldots + b_1 D + b_0$

$m \leq n$

D - oznaka za I izvod

$a_0, a_1, \ldots a_n, b_0, b_1, \ldots b_n$ - KONSTANTE

<u>SLOBODNI ODZIV</u>

- nema spoljašnji ODZIV, a IMA početni uslov

$\boxed{f(t) \equiv 0}$

$$Q(D)\,y(t) = 0 \qquad (2)$$

+ početni uslov

$$Q(\lambda) = 0$$

$$\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0$$

$\lambda_1, \lambda_2 \dots \lambda_n$ → pretpostavimo da su svi korjeni realni i svi različiti

$$y(t) = \sum_{k=1}^{n} C_k e^{\lambda_k t} \equiv y_{SLOB}(t)$$

$C_1, C_2 \dots C_n$ – iz početnih uslova

PRINUDNI ODZIV

$$Q(D)y(t) = P(D)f(t)$$

+ svi početni uslovi $= 0$

Rješenje: $y(t) \equiv y_{PRIN}(t)$

$$\boxed{y(t) = y_{PRIN}(t) + y_{SLOB}(t)}$$

↓ rješenje (1) jednačine tj. ukupni odziv (slobodni + prinudni)

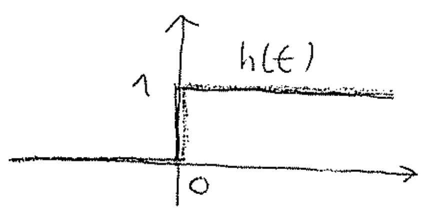
VJEŽBE

$$E = \int_{-\infty}^{+\infty} f^2(t)\,dt$$ → ENERGIJA

$$\int t^2 dt = \frac{t^3}{3}$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

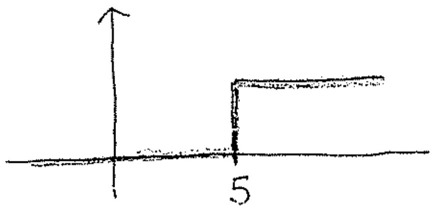
Hevisajd-ova funkcija

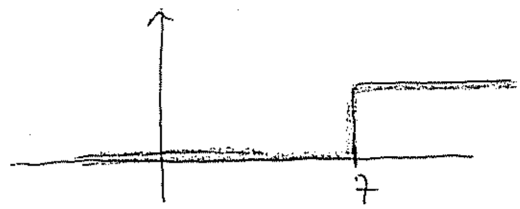


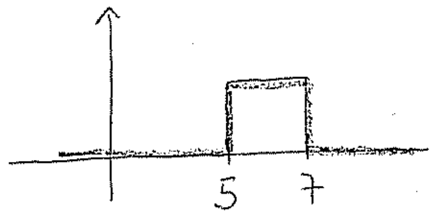
(1) Skicirajte signal

$g(t) = h(\underbrace{t-5}_{\text{raste}}) - h(t-7)$

Hevisajd-ova funkcija:



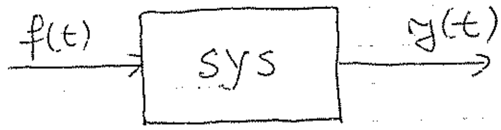




(2) LINEARNOST

Uslov linearnosti sistema se može definisati na sledeći način

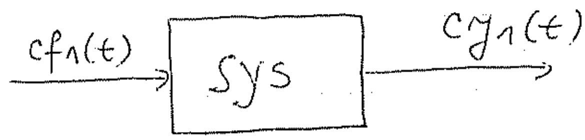
L_0



Ako je $y(t)$ odziv sistema na pobudu $f(t)$ tada je:

1° odziv na pobudu

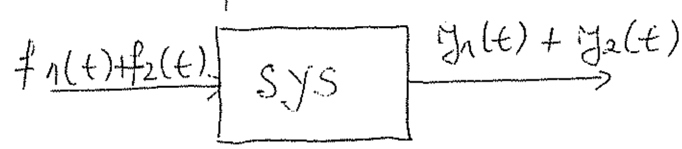
L_1



$cf_1(t) = cy_1(t)$

pri čemu je $c = \text{const}$

2° odziv na pobudu



(a) $\frac{dy}{dt} + 2y(t) = f^2(t)$

$L_{1(a)}$ $C\frac{dy}{dt}+2Cy(t)=C^2f^2(t) \quad /C\neq 0$

$$\frac{dy}{dt}+2y(t)=Cf^2(t)$$

važi ako je $C=1$, znači sistem nije linearan
strane nisu jednake
ako prvi uslov nije zadovoljen nisu ni ostali

(b) $\frac{dy}{dt}+3t\,y(t)=t^2f(t)$

L_1 $C\frac{dy}{dt}+3tCy(t)=t^2Cf(t) \quad /C$

$$\frac{dy}{dt}+3ty(t)=t^2f(t)$$ TAČNO

$$\frac{d(y_1+y_2)}{dt}+3t\left(y_1(t)+y_2(t)\right)=t^2\left(f_1(t)+f_2(t)\right)$$

$$\frac{dy_1}{dt}+3ty_1(t)+\frac{dy_2}{dt}+3\left(y_2(t)\right)=$$

$$t^2\left(f_1(t)+f_2(t)\right)$$

$$t^2f_1(t)+t^2f_2(t)\equiv t^2\left(f_1(t)+f_2(t)\right)$$ Tačno

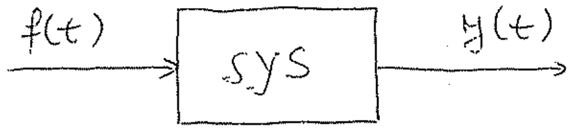
Linearan je

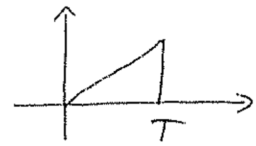
(c) $\frac{dy}{dt}+y^2(t)=f(t)$ nije linearan

(d) $3y(t)+2=f(t)$ nije linearan

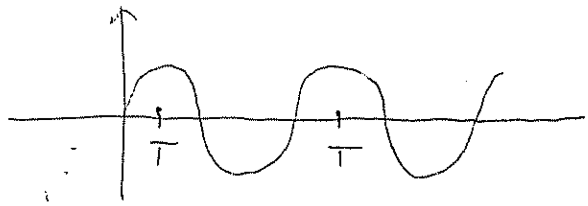
(3) INVARIJANTNOST

Sistem je vremenski invarijantan ako iz pretpostavke da je $y(t)$ odziv na signal $f(t)$ slijedi da je $y_1(t) = y(t-T)$. Odziv na pobudu $f_1(t) = f(t-T)$ za svaku vrijednost vremenskog pomjeraja T.





$$f_1(t) = f(t-T) \qquad y_1(t) = y(t-T)$$



ⓐ $y(t) = f(t-2)$

$$y_1(t) = f_1(t-2) \underset{\Delta}{=} f((t-2)-T) = f(t-T-2)$$

$$f_1(t) = f(t-T)$$

ovdje smo dokazivali da li je $y_1(t) = f(t-T)$

$$y(t-T) = f(t-2-T) = f(t-T-2)$$

ovdje smo dokazivali da li je $y(t-T) = f_1(t)$

znači sistem je vremenski invarijantan

ⓑ $y(\underbrace{t}_{t-T}) = f(\underbrace{-t}_{t-T})$

$-T \qquad -T$

Dokazujemo da li je $y_1(t) = f(t-T)$

$$y_1(t) = f_1(-t) \underset{\Delta}{=} f(-t-T) \qquad (1)$$

$$f_1(t) = f(t-T)$$

$$y(t-T) = f(-\underbrace{(t-T)}_{t}) = f(-t+T) \quad (2)$$

(1) i (2) su različiti i sistem nije vremenski invarija:

$$y(t) = f(-t)$$
$$\wedge \qquad\quad \wedge$$
$$t-T \qquad t-T$$

$$y(t-T) = f(-(t-T))$$

c) $y(t) = t f(t-2)$ nije vremenski invarijantan

$$\boxed{\text{PREDAVANJE}}$$

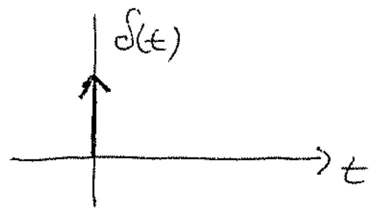
IMPULSNI ODZIV SISTEMA

$f(t)=\delta(t) \longrightarrow$ SISTEM svi početni uslovi $=0$ $\longrightarrow y(t) \equiv y_{prin}(t) \equiv g(t)$ impulsni odziv

$$\boxed{\begin{array}{c} Q(D)\, g(t) = P(D)\, \delta(t) \\ \text{svi početni uslovi} = 0 \end{array}} \qquad (1)$$

$$\delta(t) \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$



Pojam impulsnog odziva je važan iz sledećeg razloga. Ako je poznat taj impulsni odziv tj. ako je nađena funkcija $g(t)$ iz jednačine (1) onda se može naći odziv sistema za proizvoljnu pobudu $f(t)$.

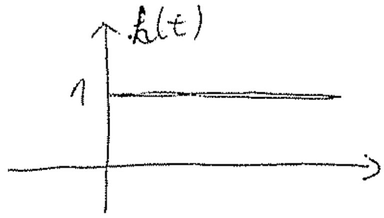
Rješenje jednačine (1) bez izvođenja ima sledeći oblik:

$$\boxed{g(t) = b_1 \delta(t) + \left[P(D)\, g_n(t)\right] h(t)}$$

Konstanta b_n postoji samo ako je $m=n$.
Za sve ostale slučajeve $b_n=0$ kada je $m<n$. Tada rješenje ima prostiji oblik:

$$\boxed{g(t) = \left[P(D)\, g_n(t)\right] h(t)}$$

$$h(t) = \begin{cases} 1 & t>0 \\ 0 & t<0 \end{cases}$$



$g(t)$ postoji samo za $t<0$ $\quad y(t)=0$
Funkcija $g(t)$ dobija se iz iste jednačine kao i slobodni odziv sistema.

$$Q(D)\, g_n(t) = 0$$

$$\boxed{g_n(t) = \sum_{k=1}^{n} C_k e^{\lambda_k t}}$$

$$\left.\begin{array}{l} n-1 \begin{cases} g_n(0) = 0 \\ g_n'(0) = 0 \\ \vdots \\ g_n^{(n-2)}(0) = 0 \end{cases} \\ n \begin{cases} g_n^{(n-1)}(0) = 1 \end{cases} \end{array}\right\}$$

odavde izračunavamo $C_1, C_2 \ldots C_n$ i tako ih zadajemo

PRIMJER

Sistem je opisan diferencijalnom jednačinom:

$$(D^2+3D+2)\underbrace{y(t)}_{\text{odziv}} = \underbrace{Df(t)}_{\text{pobuda}}$$

Naći impulsni odziv tj. funkciju $g(t)=?$

$$\left.\begin{array}{l}(D^2+3D+2)\,g(t) = D\delta(t) \\ \text{svi početni uslovi} = 0\end{array}\right\}$$

$Q(D) = D^2+3D+2$ $\quad n=2$

$P(D) = D$ $\quad m=1$

pošto je $m<n$ ide jednačina:

$$g(t) = [P(D)\,g_n(t)]\,h(t)$$

$$\lambda^2+3\lambda+2=0$$

$$\lambda_1=-1$$

$$\lambda_2=-2$$

$$g_n(t) = C_1e^{-t}+C_2e^{-2t}$$

$$\begin{cases} g_n(0)=0 \\ g_n'(0)=1\end{cases}$$

$t=0 \quad g(0) = \boxed{0=C_1+C_2} \quad (1)$

$$g_n'(t) = -C_1e^{-t}-2C_2e^{-2t}$$

$t=0 \quad g_n'(0) = \boxed{1=-C_1-2C_2} \quad (2)$

$$\left.\begin{array}{l}C_2=-1\\ C_1=1\end{array}\right\}$$

zamjenimo u funkciju i dobijemo rješenje:

$$\boxed{g_n(t)=e^{-t}-e^{-2t}}$$

$D f(t)$

$P(D)=D$

$$P(D)\,g_n(t) = D g_n(t)$$
$$= \frac{d}{dt}\left[g_n(t)\right] =$$
$$= \frac{d}{dt}\left[e^{-t} - e^{-2t}\right] =$$
$$= -e^{-t} + 2e^{-2t}$$

$$\boxed{g(t) = \left(-e^{-t} + 2e^{-2t}\right) h(t)}$$ → impulsni odziv rješenje!

PRIMJER

Naći impulsni odziv sistema koji je opisan diferencijalnom jednačinom:

$$(D^2 + 4D + 3)\,g(t) = f(t)$$

naći $g(t) = ?$

$P(D) = 1$

$n = \underset{2}{1}$, $m = 0$ (ide formula $y(t) = g_n(t)\,h(t)$

Rješenje $\quad g(t) = \frac{1}{2}\left(e^{-t} - e^{-3t}\right) h(t)$

NALAŽENJE PRINUDNOG ODZIVA SISTEMA ZA PROIZVOLJNU POBUDU $f(t)$ AKO JE POZNAT IMPULSNI ODZIV SISTEMA $g(t)$

$f(t)$ → [SISTEM svi početni uslovi = 0] → $y(t) \equiv y_{PRIN}(t)$

↳ TRAŽIMO

znamo i $g(t)$

$$\delta(t) \longrightarrow g(t)$$
$$\delta(t-\tau) \longrightarrow g(t-\tau)$$
$$f(t) = \int_{-\infty}^{+\infty} f(\tau)\,\delta(t-\tau)\,d\tau \longrightarrow \int_{-\infty}^{+\infty} f(\tau)\,g(t-\tau)\,d\tau$$

$$\boxed{y(t) = \int_{-\infty}^{+\infty} f(\tau)\, g(t-\tau)\, d\tau} \rightarrow \text{KONVOLUCIJA } f(t) \text{ i } g(t)$$

Oznaka za konvoluciju je zvijezda.

$$\boxed{f(t) * g(t) = \int_{-\infty}^{+\infty} f(\tau)\, g(t-\tau)\, d\tau}$$

DEFINICIJA KONVOLUCIJE

$$\boxed{y(t) \equiv y_{PRIN}(t) = f(t) * g(t)}$$

$$y_{SLOB\,\delta}(t) = \sum_{k=1}^{n} C_k e^{\lambda_k t}$$

$$Q(\lambda) = 0 \Rightarrow \lambda_1, \lambda_2, \ldots, \lambda_n$$

UKUPAN ODZIV = PRINUDNI ODZIV + SLOBODNI ODZIV

$$\boxed{y(t) = \underbrace{f(t) * g(t)}_{\substack{\text{PRINUDNI ODZIV} \\ +\text{ svi početni uslovi}=0}} + \underbrace{\sum_{k=1}^{n} C_k e^{\lambda_k t}}_{\text{SLOBODNI ODZIV}}}$$

SVOJSTVA KONVOLUCIJE

1° $f_1(t) * f_2(t) = f_2(t) * f_1(t)$

2° $f(1) * \delta(t) = \int_{-\infty}^{+\infty} f(\tau)\, \delta(t-\tau)\, d\tau = f(t)$

$$\boxed{f(t) * \delta(t) = f(t)}$$

igra neku ulogu jedinice

3° $f(t) = 0 \qquad t < 0$

$g(t) = 0 \qquad t < 0$

$$\boxed{f(t) * g(t) = 0 \qquad t < 0}$$

za $t<0$ konvolucija je 0-la

$t > 0$

$$f(t) * g(t) = \int_{-\infty}^{+\infty} f(\tau)\, g(t-\tau)\, d\tau =$$

$$\int_{-\infty}^{0} f(\tau)\, g(t-\tau)\, d\tau + \int_{0}^{+\infty} f(\tau)\, g(t-\tau)\, d\tau =$$

$$\int_{0}^{+\infty} f(\tau)\, g(t-\tau)\, d\tau$$

$t-\tau < 0 \qquad g(t-\tau)=0$

$-\tau < -t \quad / \cdot (-1)$

$\tau > t \quad g(t-\tau) = 0$

$$= \int_{0}^{t} f(\tau)\, g(t-\tau)\, d\tau + \int_{t}^{+\infty} f(\tau)\, g(t-\tau)\, d\tau$$

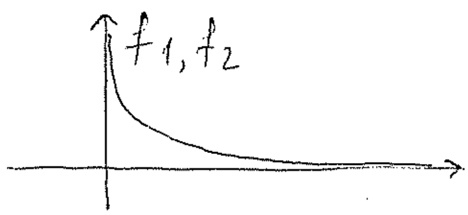
$$\boxed{f(t) * g(t) = \int_{0}^{t} f(\tau)\, g(t-\tau)\, d\tau} \quad \text{za } t>0$$

$$\boxed{f(t) * g(t) = \left(\int_{0}^{t} f(\tau)\, g(t-\tau)\, d\tau\right) h(t)} \quad \text{za } t>0$$

PRIMJER

Nađi konvoluciju funkcija:

$$f_1(t) = f_2(t) = e^{-t}\, h(t)$$



$$f_1(t) * f_2(t) = \underbrace{\left(\int_{0}^{t} f_1(\tau)\, f_2(t-\tau)\, d\tau\right)}_{I} h(t)$$

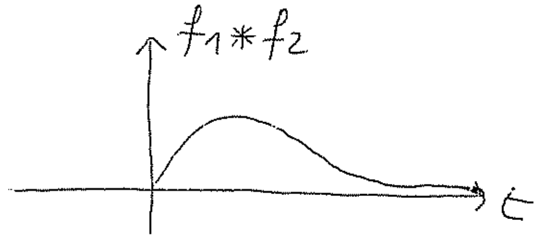
$$I=\int_0^t e^{-\tau}\cdot e^{-(t-\tau)}\,d\tau$$

$$=\int_0^t e^{-\tau}\cdot e^{-t}\cdot e^{\tau}\,d\tau$$

$$=e^{-t}\underbrace{\int_0^t d\tau}_{t}$$

$$\int_0^t d\tau=\tau\Big|_0^t=t-0=t$$

$$=t\,e^{-t}$$



ZA VJEŽBU

Naći konvoluciju funkcija

$f_1(t)=h(t)$

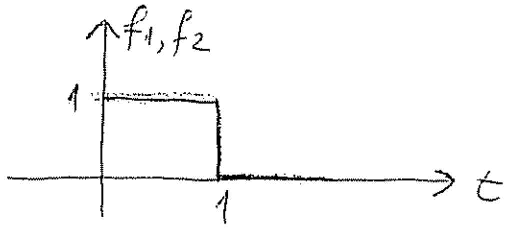
$f_2(t)=e^{-t}h(t)$

Rješenje $\quad f_1 * f_2=(1-e^{-t})\,h(t)$

PRIMJER

Naći konvoluciju ove dvije funkcije

$$f_1(t)=f_2(t)=\begin{cases}1 & 0<t<1\\ 0 & \text{za ostalo } t\end{cases}$$



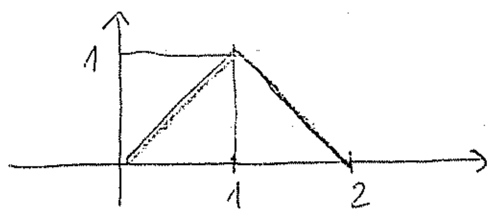
CRTAMO FUNKCIJU U $0<t<1$

-||- $1<t<2$

-||- $t>2$

REZULTAT

$$f_1 * f_2 = \begin{cases} 0 & t<0 \\ t & 0<t<1 \\ 2-t & 1<t<2 \\ 0 & t>2 \end{cases}$$



Šta se dešava ako je pobuda eksponencijalna funkcija
$f(t)=e^{st}$

tražimo prinudni odziv

$$y(t) \equiv y_{prin}(t) = f(t) * g(t) = g(t) * f(t) =$$

$$\int_{-\infty}^{+\infty} e^{st} g(t-\tau)\, d\tau = \int_{-\infty}^{+\infty} g(\tau) \underbrace{f(t-\tau)}_{e^{s(t-\tau)}} d\tau =$$

$$\int_{-\infty}^{+\infty} g(\tau) e^{s(t-\tau)} d\tau = \int_{-\infty}^{+\infty} g(\tau) e^{st} e^{-s\tau} d\tau =$$

$$e^{st} \underbrace{\int_{-\infty}^{+\infty} g(\tau) e^{-s\tau} d\tau}_{\text{konstanta ako je } s=\text{const}}$$

$$\boxed{y(t) \equiv y_{prin}(t) = \text{const} \cdot e^{st}}$$

Za ovaj slučaj odziv i pobuda se poklapaju.

U opštem slučaju $y(t)$ prinudni odziv biće neka funkcija s

$$\boxed{y(t) \equiv y_{prin}(t) = F(s) \cdot e^{st}}$$

PRIMJER

Sistem je opisan diferencijalnom jednačinom II reda. Glasi:

$$(D^2 + 3D + 2)\, y(t) = f(t)$$

$$f(t) = 2e^{-3t} h(t)$$

početni uslovi su:

$$y(0) = 0$$

$$y'(0) = -1$$

SLOBODNI ODZIV

$$(D^2 + 3D + 2)\, y(t) = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\lambda_1 = -1 \qquad \lambda_2 = -2$$

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = C_1 e^{-t} + C_2 e^{-2t}$$

za $t = 0$ $\quad y(0) = \boxed{0 = C_1 + C_2}$

$$y'(t) = -C_1 e^{-t} - 2C_2 e^{-2t}$$

$$y'(0) = \boxed{-1 = -C_1 - 2C_2}$$

$$\boxed{C_2 = 1} \qquad \boxed{C_1 = -1}$$

$$y(t) = \left(-e^{-t} + e^{-2t}\right) h(t) \equiv y_{slob}(t)$$

$$y_{prin}(t) = f(t) * g(t)$$

$$g(t) = ?$$

$$g(t) = [\underbrace{P(D)}_{\equiv 1} g_n(t)]\, h(t) = g_n(t)\, h(t)$$

$$g_n(t) = Ae^{-t} + Be^{-2t}$$

$g_n(0) = 0$ $\qquad$ $0 = A + B$

$g_n'(0) = 1$ $\qquad$ $g_n'(t) = -Ae^{-t} - 2Be^{-2t}$

$$\boxed{1 = -A - 2B}$$

$$\boxed{B = -1}$$
$$\boxed{A = 1}$$

$$\boxed{g(t) = \left(e^{-t} - e^{-2t}\right) h(t)} \to \text{rješenje}$$

$$y_{prin}(t) = f(t) * g(t) = \left(\int_0^t 2e^{-3\tau}\left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right) d\tau\right) h(t)$$

$$= \int_0^t 2e^{-3\tau} e^{-t} e^{-\tau} d\tau$$

$$= 2e^{-t} \int_0^t e^{-2\tau} d\tau$$

$$= \int e^{m\tau} d\tau = \frac{1}{m} e^{m\tau} \Big|_0^t$$

$$= \frac{1}{m} e^{m\tau} - \frac{1}{m}$$

$$\boxed{y_{prin}(t) = \left(e^{-t} - 2e^{-2t} + e^{-3t}\right) h(t)} \to \text{REZULTAT}$$

<u>PROVJERA</u>

$$y_{prin}(0) = y'_{prin}(0) = 0$$

$$y_{prin}(0) = 0$$

$$y'_{prin}(t) = \left(-e^{-t} + 4e^{-2t} - 3e^{-3t}\right) h(t)$$

$$y'_{prin}(0) = -1 + 4 - 3 = 0$$

Ukupan odziv (zbir ova dva odziva)

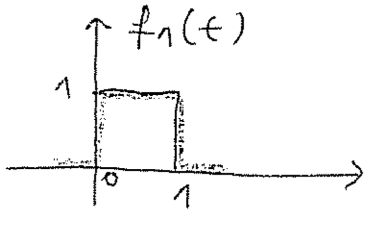
$$y(t) = y_{prin}(t) + y_{slob}(t)$$
$$= (e^{-t} - 2e^{-2t} + e^{-3t})h(t) + (-e^{-t} + e^{-2t})h(t)$$
$$= (-e^{-2t} + e^{-3t})h(t)$$

$\boxed{\text{VJEŽBE}}$

KONVOLUCIJA

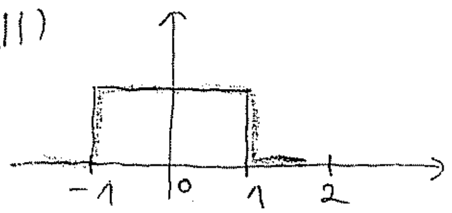
$$f(t) * g(t) = \int_{-\infty}^{+\infty} f(\tau)\, g(t-\tau)\, d\tau$$

(I)

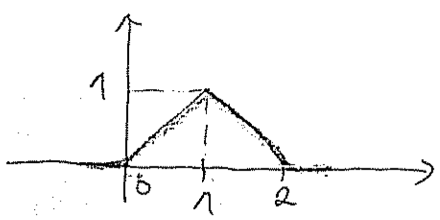


$$\int_{-\infty}^{\infty} f_1(\tau)\, f_1(t-\tau)\, d\tau$$

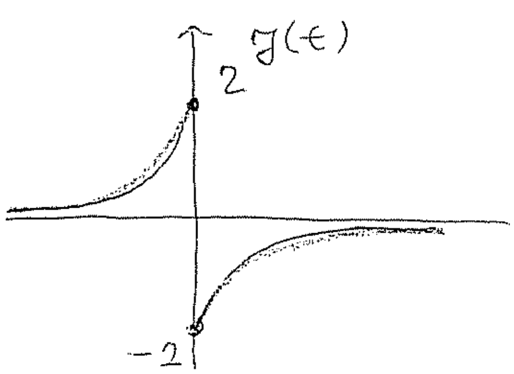
(II)

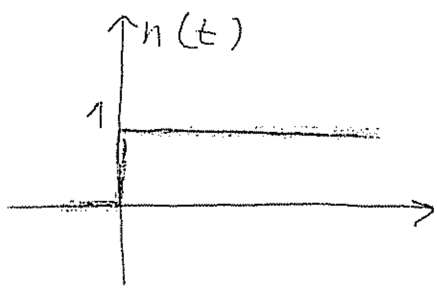


(III)



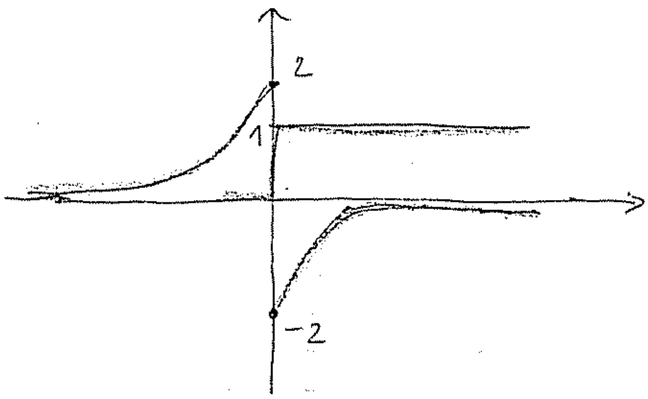
PRIMJER



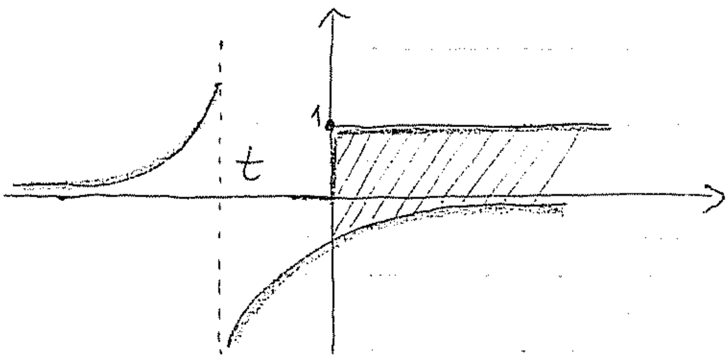


$$\begin{array}{l} z \\ \check{z} \end{array}\begin{cases} 2e^{-t} & t>0 \\ -2e^{2t} & t<0 \end{cases}$$

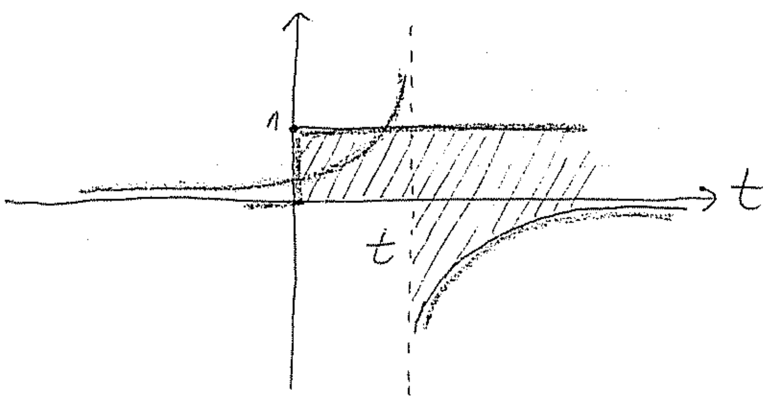
$$\boxed{\int e^{at}\,dt = \frac{1}{a}e^{at}}$$



nacrtamo inverznu funkciju +



$t<0$



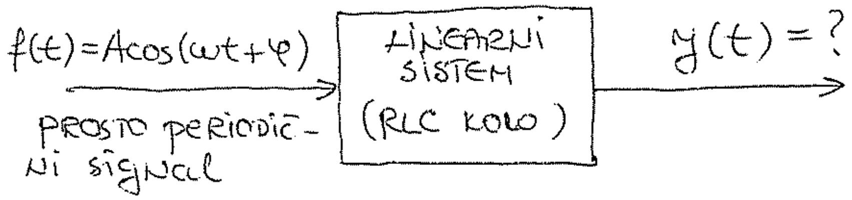
$t>0$

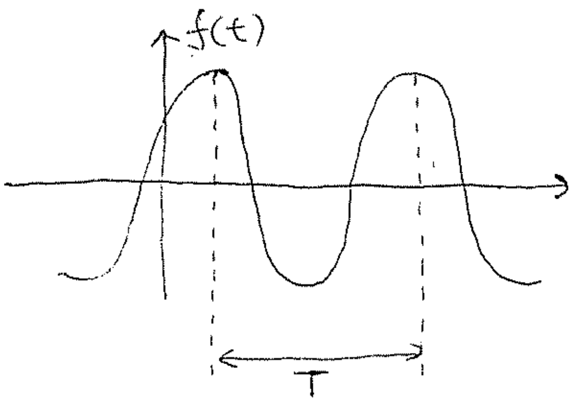
$$\int_0^t \check{z}\,dt + \int_t^{-\infty} z\,dt$$

PREDAVANJE

PREDSTAVLJANJE SLOŽENOPERIODIČNOG SIGNALA U VIDU SUME PROSTOPERIODIČNIH SIGNALA

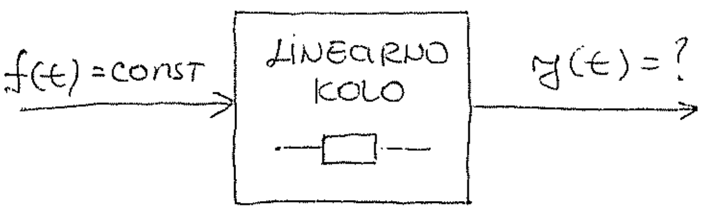
FURIJE-OVA ANALIZA SIGNALA

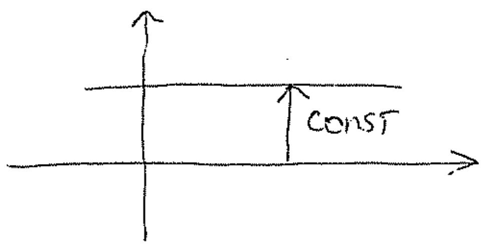




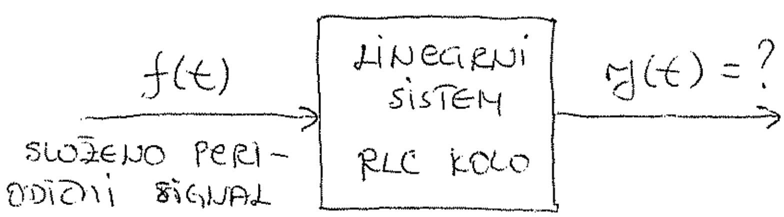
$$\boxed{\omega_0=\frac{2\pi}{T}}$$

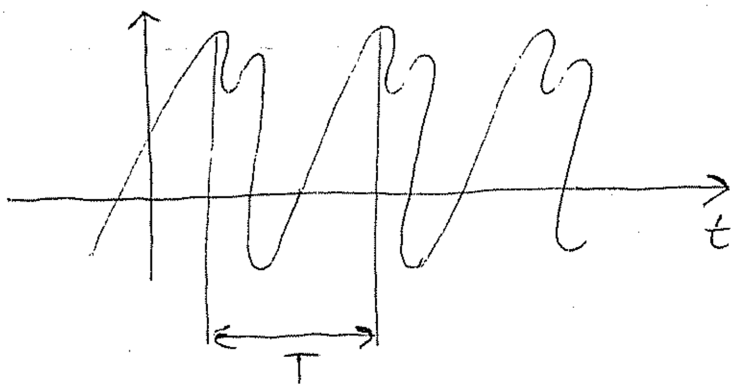
$$\boxed{T=\frac{2\pi}{\omega_0}}$$



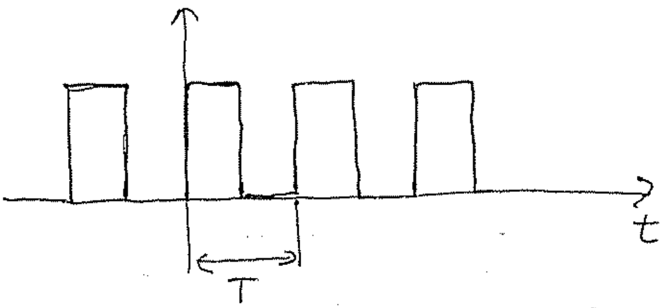


$$\boxed{\omega_0=0}$$





NIJE PERIODIČAN



JESTE PERIODIČAN

$$\boxed{f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \varphi_n)} \quad (1)$$

$$= C_0 + C_1 \cos(\omega_0 t + \varphi_1) + C_2 \cos(2\omega_0 t + \varphi_2) + \dots$$

$$C_0 \rightarrow y_0$$

$$C_1 \cos(\omega_0 t + \varphi_1) \rightarrow y_1$$

$$C_2 \cos(2\omega_0 t + \varphi_1) \rightarrow y_2$$

$$f(t) \rightarrow y_0 + y_1(t) + y_2(t) + \dots$$

Svaki složeni periodični signal može se predstaviti u vidu u opštem slučaju beskonačne sume prostoperiodičnih signala. (1)

$$\omega_0 = \frac{2\pi}{T}$$

C_0, C_1, C_2, C_3 – treba naći

Konstantan član C_0 se naziva NULTI HARMONIK.

Konstantan član $C_1 \cos(\omega_0 t + \varphi_1)$ je PRVI i OSNOVNI HARMONIK, njegova perioda je jednaka periodi zadanih

složenoperiodičnih signala.
Konstantan član $C_2\cos(\omega_0 t+\varphi_2)$ je DRUGI HARMONIK.

$$f(t)=C_0+\sum_{n=1}^{\infty} C_n\cos(n\omega_0 t+\varphi_n)$$

$$f(t)=C_0+\sum_{n=1}^{\infty}\left(\overbrace{C_n\cos n\omega_0 t\cdot\cos\varphi_n}^{a_1}-\overbrace{C_n\sin n\omega_0 t\cdot\sin\varphi_n}^{b_1}\right)$$

$a_0=C_0$

$a_n=C_n\cos\varphi_n$

$b_n=C_n\sin\varphi_n$

$$\boxed{f(t)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos n\omega_0 t+b_n\sin n\omega_0 t\right)}\quad(2)$$

$a_0=?$

$a_n=?\quad n=1,2,3\ldots$

$b_n=?\quad n=1,2,3\ldots$

Relacija (2) se pomnoži sa nekom konstantom i integral po bilo kom intervalu dužine T.

$(t_1\rightarrow t_1+T)\qquad t_1\rightarrow$ proizvoljno

$$\int_{t_1}^{t_1+T}K\cdot f(t)\,dt=\int_{t_1}^{t_1+T}a_0\cdot K\,dt+\sum_{n=1}^{\infty}K a_n\overbrace{\int_{t_1}^{t_1+T}\cos n\omega_0 t\,dt}^{0}+$$

$$+\sum_{n=1}^{\infty}K b_n\overbrace{\int_{t_1}^{t_1+T}\sin n\omega_0 t\,dt}^{0}=$$

$$=\cancel{K}\int_{t_1}^{t_1+T}f(t)\,dt=a_0\cancel{K}T$$

integrali sin i cos su 0-le.

$$\int_{t_1}^{t_1+T} \cos n\omega_0 t \, dt = 0 \qquad \int_{t_1}^{t_1+T} \sin n\omega_0 t \, dt = 0$$

dobijamo formulu:

$$\boxed{a_0 = \frac{1}{T}\int_{t_1}^{t_1+T} f(t)\,dt} \qquad t_1 \text{ proizvoljno}$$

Relacija (2) se pomnoži sa funkcijom $\cos m\omega_0 t$ i integralom $\int_{t_1}^{t_1+T}$

$$\int_{t_1}^{t_1+T} f(t)\cos m\omega_0 t\,dt = \int_{t_1}^{t_1+T} \omega_0 \cancel{\cos m\omega_0 t}^{\,0}\,dt +$$

$$+\sum_{n=1}^{\infty} a_n \int_{t_1}^{t_1+T} \cos n\omega_0 t \cos m\omega_0 t\,dt + \sum_{n=1}^{T} b_n \int_{t_1}^{t_1+T} \sin n\omega_0 t \cos m\omega_0 t\,dt =$$

$$= \int_{t_1}^{t_1+T} f(t)\cos m\omega_0 t\,dt = a_m \cdot \frac{T}{2}$$

$$a_m = \frac{2}{T}\int_{t_1}^{t_1+T} f(t)\cos m\omega_0 t\,dt$$

$$\boxed{a_n = \frac{2}{T}\int_{t_1}^{t_1+T} f(t)\cos n\omega_0 t\,dt} \qquad \begin{array}{l} m \to n \\ n = 1, 2, 3 \dots \end{array}$$

<u>važi za:</u>

$$\int_{t_1}^{t_1+T} \cos n\omega_0 t \cos m\omega_0 t \, dt = \begin{cases} 0 & n \neq m \\ T/2 & n = m \end{cases}$$

$$\int_{t_1}^{t_1+T} \sin n\omega_0 t \cos m\omega_0 t \, dt = 0$$

Ako se relacija (2) pomnoži sa $\sin m\omega_0 t$ dobija se:

$$\boxed{b_n = \frac{2}{T} \int_{t_1}^{t_1+T} f(t) \sin n\omega_0 t \, dt}$$

Red (odnosno beskonačna suma u relaciji (2)) naziva se Furijerov red funkcije $f(t)$.

$C_0, C_n, \varphi_n \quad n = 1, 2, 3 \dots$

$$\boxed{C_0 = a_0}$$

$$\left.\begin{aligned} a_n &= C_n \cos\varphi_n \;|^2 \\ b_n &= -C_n \sin\varphi_n \;|^2 \end{aligned}\right] +$$

$$a_n^2 + b_n^2 = C_n^2 \cos^2\varphi_n + C_n^2 \sin^2\varphi_n = C_n^2$$

$$\boxed{C_n = \sqrt{a_n^2 + b_n^2}}$$

$$\frac{b_n}{a_n} = -\operatorname{tg}\varphi_n$$

$$\operatorname{tg}\varphi_n = -\frac{b_n}{a_n}$$

$$\varphi_n = \mathrm{arc\,tg}\left(-\frac{b_n}{a_n}\right)$$

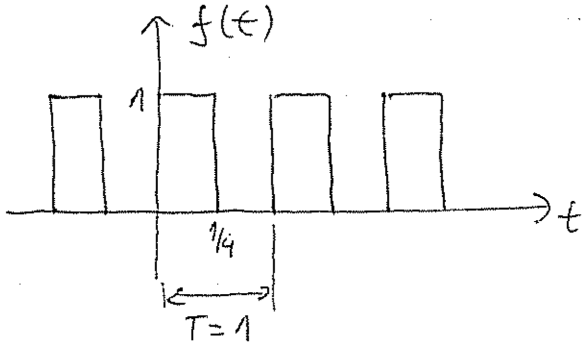
$a_n \rightarrow$

$b_n \rightarrow 0$ kad $n \rightarrow \infty$

$$C_n = \sqrt{a_n^2 + b_n^2} \rightarrow 0 \qquad n \rightarrow +\infty$$

PRIMJER

Dat je složenoperiodični signal kao na slici. Predstaviti ga u vidu sume složenoperiodičnog signala (razviti funkciju $f(t)$ u Furijerov red to je isto)



$$\omega_0 = \frac{2\pi}{T} = 2\pi$$

$$f(t) = \begin{cases} 1 & 0 < t < 1/4 \\ 0 & t_1 < t < 1 \end{cases}$$

$$\begin{cases} f(t+1) = f(t) \quad \forall t \\ f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \end{cases}$$

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt = \int_0^1 f(t)\,dt = \int_1^{1/4} 1\,dt + \int_{1/4}^1 \cancel{0\,dt}^{\,0}$$

$$\boxed{a_0 = \frac{1}{4}}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\, dt$$

$$= 2\int_0^1 f(t)\cos n\omega_0 t\, dt$$

$$= 2\int_0^{1/4} \cos n\omega_0 t\, dt$$

$$= 2\,\frac{1}{n\omega_0}\sin n\omega_0 t\,\Big|_0^{1/4}$$

$$= \frac{2}{n\omega_0}\sin\frac{n\omega_0}{4} - \cancel{\frac{2}{n\omega_0}\sin 0}^{\,0}$$

$$= \frac{2}{n\omega_0}\sin\frac{n\omega_0}{4}$$

$$= \frac{2}{n2\pi}\sin\frac{2n\pi}{4}$$

$$\boxed{a_n = \frac{1}{n\pi}\sin\frac{n\pi}{2}}$$

$$\boxed{\begin{aligned}\int \cos mx\,dx &= \frac{1}{m}\sin mx\\ \int \sin mx\,dx &= -\frac{1}{m}\cos mx\end{aligned}}$$

PROVJERA za a_n:

$a_1 = 1/\pi$

$a_2 = 0$

$a_3 = -\frac{1}{3\pi}$

$\vdots$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\,dt$$

$$= 2\int_0^1 f(t)\sin n\omega_0 t\,dt$$

$$= 2\int_0^{1/4}\sin n\omega_0 t\,dt$$

$$= -2\frac{1}{n\omega_0}\cos n\omega_0 t\Big|_0^{1/4}$$

$$= \frac{2}{n\omega_0}\left(1-\cos\frac{n\omega_0}{4}\right)$$

$$\boxed{b_n = \frac{1}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)}$$

PROVJERA ZA b_n:

$$b_1 = \frac{1}{\pi}$$

$$b_2 = \frac{1}{\pi}$$

$$b_3 = \frac{1}{3\pi}$$

$\vdots$

$$C_0 = a_0 = \frac{1}{4}$$

$$C_n = \sqrt{a_n^2 + b_n^2}$$

$$\left.\begin{aligned} C_1 &= \sqrt{2}/\pi \\ C_2 &= 1/\pi \\ C_3 &= \sqrt{2}/3\pi \end{aligned}\right\} \text{RJEŠENJA} \qquad \left.\begin{aligned} \varphi_1 &= -\pi/4 \\ \varphi_2 &= -\pi/2 \\ \varphi_3 &= -3\pi/4 \end{aligned}\right\} \text{Rješenja}$$

$\vdots$

Ako je funkcija $f(t)$ parna:

simetrična u odnosu na y-osu

$$f(-t) = f(t)$$

$\boxed{b_n = 0}$ → za parnu funkciju

$$\boxed{f(t) = a_0 + \sum_{1}^{\infty} a_n \cos n\omega_0 t}$$ → za parnu funkciju

Ako je funkcija $f(t)$ neparna:

simetrična na koordinatni početak

$$f(-t) = -f(t)$$

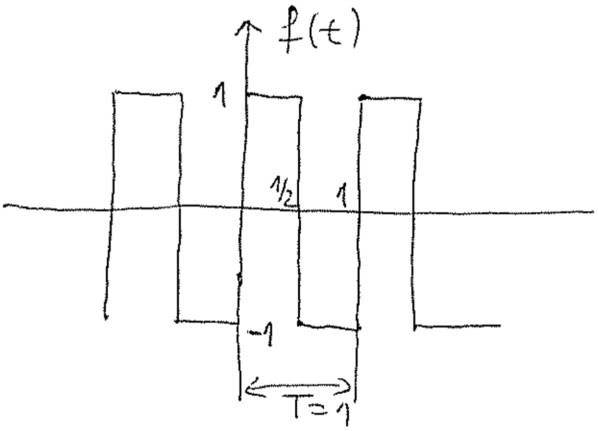
$\boxed{a_0 = 0}$ → za neparnu funkciju

$a_n = 0 \qquad n = 1, 2, 3 \ldots$

$$\boxed{f(t) = \sum_{1}^{\infty} b_n \sin n\omega_0 t}$$ → za neparnu funkciju

<u>ZADATAK ZA VJEŽBU:</u>

Razviti Furijerov red, složenoperiodični signal prikazan na slici



$$\omega_0 = \frac{2\pi}{T} = 2\pi$$

Funkcija je neparna

<u>REZULTAT</u>

$$f(t) = \sum_{n=1,3,5}^{\infty} \frac{4}{n\pi} \underbrace{\sin n\omega_0 t}_{\approx \cos\left(n\omega_0 t - \frac{\pi}{2}\right)}$$

$b_n = \frac{4}{n\pi} \quad n = 1,3,5\ldots$

$b_1 = \frac{4}{\pi}$

$b_2 = 0$

$b_3 = \frac{4}{3\pi}$

$b_4 = 0$

$b_5 = \frac{4}{5\pi}$

$\vdots$

$\cancel{C_0} + \sum C_n \cos(n\omega_0 t + \varphi_n)$

$C_n = \sqrt{\cancel{a_n^2} + b_n^2} = b_n \quad n = 1,3,5\ldots$

$\varphi_n = -\frac{\pi}{2} \quad n = 1,3,5\ldots$

FURIJEROV SPEKTAR SLOŽENOPERIODIČNOG SIGNALA

$f(t) = C_0 + \sum C_n \cos(n\omega_0 t + \varphi_n)$

– je skup svih vrijednosti C_0, C_n i φ_n. Dijeli se na dva spektra:

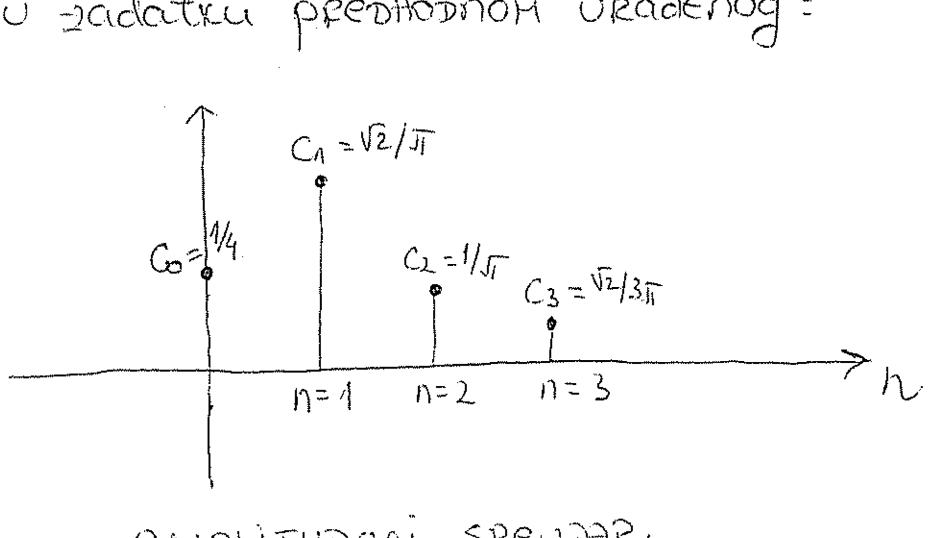
1° AMPLITUDNI SPEKTAR $\{C_0, C_n\}$ $n = 1,2,3\ldots$

2° FAZNI SPEKTAR $\{\varphi_n\}$ $n = 1,2,3\ldots$

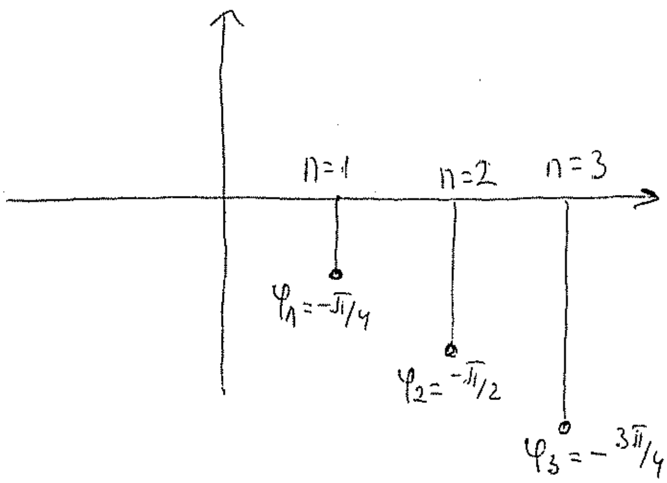
Ova dva spektra se obično prikazuju grafički.

PRIMJER:

Skicirati amplitudni i fazni spektar signala $f(t)$ datog u zadatku prethodnom urađenog:



AMPLITUDSKI SPEKTAR

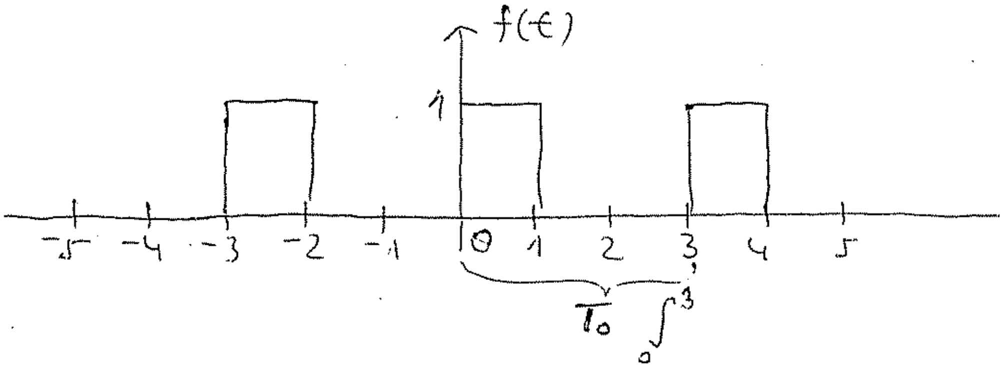


ZA VJEŽBU ZADATAK:

Skicirati amplitudski i fazni spektar u zadatku za vježbu.

$\boxed{\text{VJEŽBE}}$

① Odrediti koeficijente C_n razvoja u kompleksni Furijerov red signala $f(t)$ datog na slici:



$$T_0 = 3$$

$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{3}$$

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}}$$

$$C_n = \frac{1}{T_0}\int_0^{T_0} f(t)\, e^{-jn\omega_0 t}\, dt =$$

$$\frac{1}{3}\int_0^3 1\cdot e^{-jn\frac{2\pi}{3}t}\, dt =$$

$$\frac{1}{3}\int_0^1 e^{-jn\frac{2\pi}{3}t}\, dt =$$

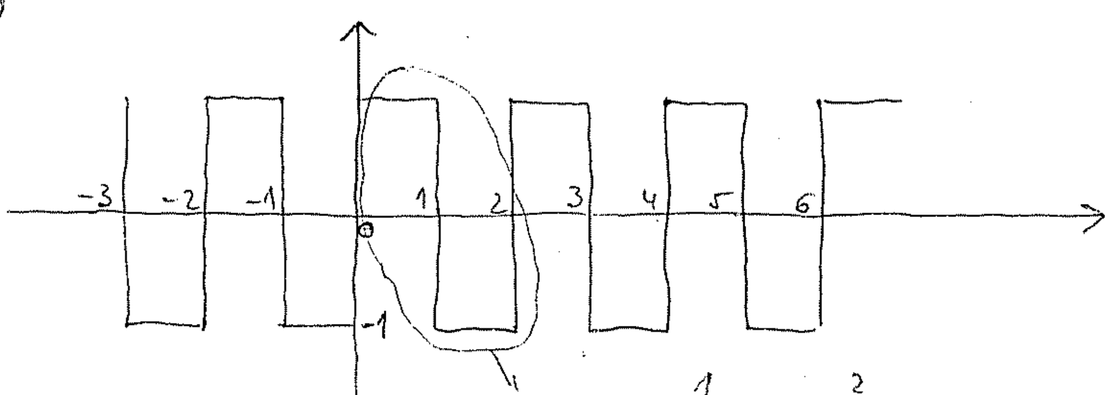
$$-\frac{1}{3jn\frac{2\pi}{3}}\cdot e^{-jn\frac{2\pi}{3t}}\Big|_0^1 =$$

$$\frac{j}{2\pi n}\left[e^{-j\frac{2n\pi}{3}} - 1\right] \quad n \neq 0$$

$$C_0 = \frac{1}{3}\int_0^3 f(t)\, dt = \frac{1}{3}\int_0^1 dt = \frac{1}{3} \quad n \neq 0$$

$$\boxed{e^{\underbrace{-jn\frac{2\pi}{3}}_{\text{konstanta}}t} \Rightarrow \frac{1}{-jn\frac{2\pi}{3}}\cdot e^{-jn\frac{2\pi}{3}t}} \qquad \boxed{j = -\frac{1}{j}}$$

(2)



To su: $\int_0^1 (1)$, $\int_0^2 (-1)$

$T = 2$

$\omega_0 = \frac{2\pi}{2} = \pi$

$$C_n = \frac{1}{2}\int_0^2 f(t)\, e^{-jn\omega_0 t}\, dt =$$

$$\frac{1}{2}\left[\int_0^1 1\, \underbrace{e^{-jn\pi t}}_{\text{const.}}\, dt + \int_0^2 (-1)\, e^{-jn\pi t}\, dt\right] =$$

$$\frac{1}{-jn\pi} e^{-jn\pi t}$$

$$\frac{1}{2}\left(-\frac{1}{jn\pi} e^{-jn\pi t}\Big|_0^1 + \frac{1}{jn\pi} e^{-jn\pi t}\Big|_1^2\right) =$$

$$2\frac{1}{2n\pi j}\left(1 - e^{-jn\pi}\right) = \frac{1}{n\pi j}\left(1 - (-1)^n\right) =$$

$$\begin{cases} \frac{1}{n\pi j}(1-1) = \emptyset & n \to \text{parno} \\ \frac{1}{4\pi j}(1-(-1)) = \frac{2}{n\pi j} & n \to \text{neparno} \end{cases}$$

$$\int e^{-jn\pi t}\, dt = -\frac{1}{jn\pi} e^{-jn\pi t}$$

$e^{-j2n\pi} = 1$ $\Rightarrow$ za parne brojeve

$e^{-jn\pi} = (-1)^n$ $\Rightarrow$ ako je n neparno onda je $(-1)^n$, a ako je n parno onda je 1-ca

$$\boxed{\begin{aligned}&\cos 2n\pi = 1\\&\sin 2n\pi = 0\\&\cos(n\pi) = (-1)^n\\&\sin(n\pi) = 0\\&e^{-j2n\pi} = 1\\&e^{-jn\pi} = (-1)^n\end{aligned}}$$

PREDAVANJE

EKSPONENCIJALNI (KOMPLEKSNI) OBLIK FURIJEROVOG REDA

(1) $f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$

$$a_n = \frac{2}{T} \int_T f(t) \cos n\omega_0 t \, dt$$

$$b_n = \frac{2}{T} \int f(t) \sin n\omega_0 t \, dt$$

ako je:

$n \longrightarrow -n$

$a_{-n} = a_n$

$b_{-n} = -b_n$

(2) $f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \varphi_n)$

$C_0 = a_0 \qquad C_n = \sqrt{a_n^2 + b_n^2}$

$\varphi_n = \operatorname{arctg}\left(\frac{-b_n}{a_n}\right)$

$$\boxed{\begin{array}{l} \text{ako je:} \\ n \longrightarrow -n \\ C_{-n} = C_n \\ \varphi_{-n} = -\varphi_n \end{array}}$$

U formi (1) iskoristićemo formule:

$$\cos x = \frac{e^{jx} + e^{-jx}}{2}$$

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

$$\boxed{\begin{array}{l} e^{jx} = \cos x + j\sin x \\ e^{-jx} = \cos x - j\sin x \end{array}}$$

$$f(x) = \underbrace{a_0}_{D_0} + \sum_{n=1}^{\infty} \Big(\underbrace{\frac{a_n - jb_n}{2}}_{\underline{D}_n}\Big) e^{jn\omega_0 t} + \Big(\underbrace{\frac{a_n - jb_n}{2}}_{\underline{D}_n^* = \underline{D}_{-n}}\Big) \cdot e^{-jn\omega_0 t}$$

crtica označava da se radi o kompleksnom broju

konjugacioni kompleks

$$f(x) = \sum_{n=-\infty}^{+\infty} \underline{D}_n e^{jn\omega_0 t}$$

$$\boxed{\underline{D}_{-n} = \frac{a_{-n} - jb_{-n}}{2} = \frac{a_n + jb_n}{2} = \underline{D}_n^*}$$

(3) $$\boxed{f(t) = \sum_{n=-\infty}^{+\infty} \underline{D}_n e^{jn\omega_0 t}}$$ → III oblik Furijerovog reda

$$\underline{D}_n = \frac{a_n - jb_n}{2} = \frac{1}{2}\left[\frac{2}{T}\int_T f(t)\cos n\omega_0 t\,dt - \frac{2}{T}j\int_T f(t)\sin n\omega_0 t\,dt\right]$$

$$\boxed{\underline{D}_n = \frac{1}{T}\int f(t)\, e^{-jn\omega_0 t}\,dt}$$

ide iz (3) formulu

$$\boxed{\cos n\omega_0 t - j\sin n\omega_0 t = e^{-jn\omega_0 t}}$$

U formuli (3) imamo:

$n=0$	0
$n=\pm 1$	ω_0
$n=\pm 2$	$2\omega_0$
$n=\pm 3$	$3\omega_0$

$\underline{D}_n$ preko (nađemo) C_n, φ_n

$D_0 = C_0$

$$\underline{D}_n = \frac{a_n - jb_n}{2} = \frac{\sqrt{a_n^2 + b_n^2}}{2} \cdot e^{j\,\mathrm{arctg}\left(-\frac{b_n}{a_n}\right)} = \frac{C_n}{2} e^{j\varphi_n}$$

za $n \neq 0$

$$\boxed{|\underline{D}_n| = D_n = \frac{C_n}{2}}$$

modul

$$\boxed{\arg \underline{D}_n = \varphi_n}$$

$n \longrightarrow -n$

$$D_{-n} = \frac{C_{-n}}{2} = \frac{C_n}{2} = D_n$$

$D_n \rightarrow$ parna funkcija od n

$\varphi_{-n} = -\varphi_n$

$\varphi_n \rightarrow$ neparna funkcija od n

Skup vrijednosti D_n kada uzima sve vrijednosti

$$\{\underline{D}_n\} \quad n = 0, \pm 1, \pm 2, \pm 3 \ldots$$

predstavljaju Furijerov spektar $f(t)$.

Skup modula ntl kompleksnih brojeva

$$\{|\underline{D}_n|\} \equiv \{D_n\}$$

predstavlja amplitudski spektar $f(t)$.

Skup argumenata kompleksnih brojeva

$$\{\arg \underline{D}_n\} \equiv \{\varphi_n\}$$

predstavlja Furijerov spektar signala $f(t)$.

PRIMJER:

STARA DEFINICIJA

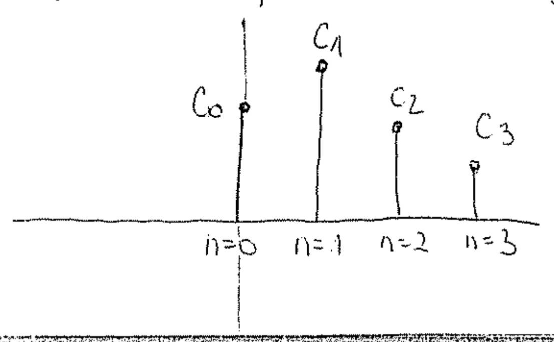
skup vrijednosti $\{C_n\}$ $n = 0, 1, 2, 3 \ldots$ je amplitudski spektar

skup vrijednosti $\{\varphi_n\}$ $n = 1, 2, 3 \ldots$ je fazni spektar

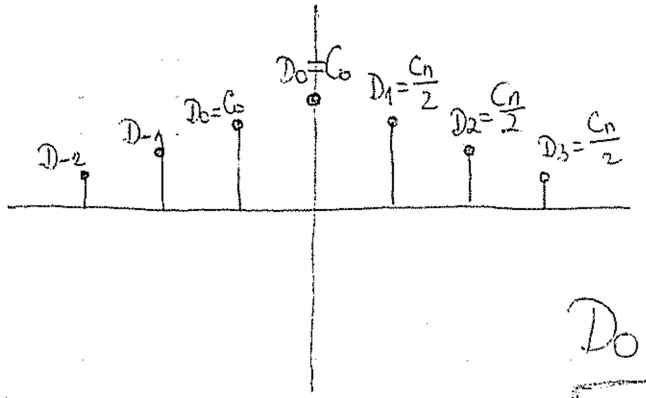
NOVA DEFINICIJA

$\{D_n\}$ $n = 0, \pm 1, \pm 2, \pm 3 \ldots$ je amplitudski spektar

$\{\varphi_n\}$ $n = \pm 1, \pm 2, \pm 3 \ldots$ je fazni spektar



amplitudski spektar po staroj definiciji

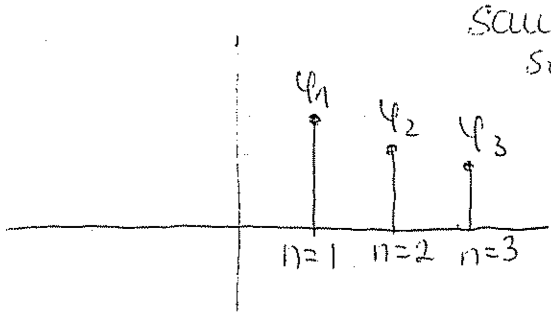


amplitudski spektar po novoj definiciji

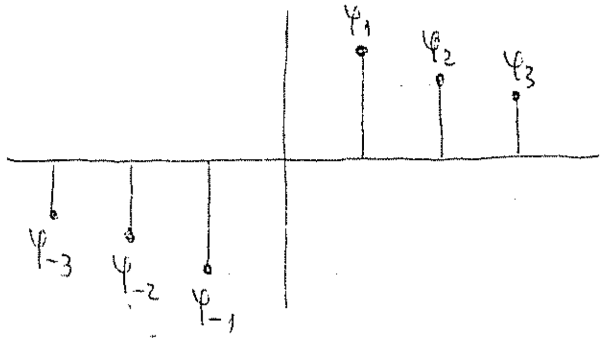
$$D_0 = C_0$$

$$\boxed{D_n = \frac{C_n}{2}}$$

samo vrijednosti sa desne strane se smanjuju na pola



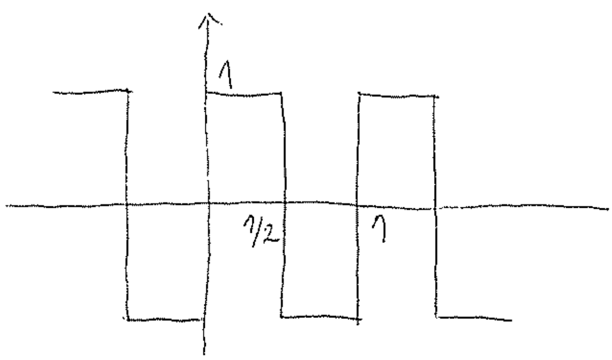
fazni spektar po staroj definiciji



fazni spektar po novoj definiciji

ZADATAK ZA VJEŽBU

Dat je složenoperiodični signal kao na slici. Naći njegov amplitudski i fazni spektar.



$\omega_0 = 2\pi$

naći $D_n = ?$ $\quad n = 0, \pm 1, \pm 2, \pm 3.$

$\varphi_n = ?$ $\quad n = \pm 1, \pm 2, \pm 3 \ldots$

PRVO NAĆI $\underline{D}_n$ (ide uz formulu (3))

$$\underline{D}_n = j\frac{1-(-1)^n}{\pi n} \quad \text{za } n \neq 0 \begin{cases} 0 & \text{za parne vrijednosti } n \\ j\frac{2}{\pi n} & \text{za neparne vrije. } n \end{cases}$$

$$D_0 = 0 \quad \text{za } n = 0$$

$$D_n = |\underline{D}_n| = \begin{cases} 0 & \text{za } n \text{ parno} \\ \frac{2}{\pi} & \text{za } n \text{ neparno} \end{cases}$$

$$\varphi_n = \arg\{\underline{D}_n\} = \begin{cases} \text{nije definisano za } n \text{ parno} \\ \pi/2 \quad \text{za } n \text{ neparno} \\ -\pi/2 \quad \text{za neparne i negativne } n \end{cases}$$

$$\left.\begin{matrix} n \\ -n \end{matrix}\right\} n\omega_0$$

$n \longrightarrow n\omega_0$

$n \longrightarrow -n\omega_0$

$n = 0$

$$n = 1 < \begin{matrix} \omega_0 \\ -\omega_0 \end{matrix}$$

$$n = 2 < \begin{matrix} 2\omega_0 \\ -2\omega_0 \end{matrix}$$

ODZIV SISTEMA (PRINUDNI) NA SLOŽENU PERIODIČNU POBUDU

pobuda $f(t)$	odziv $y(t)$
e^{st}	$G(s)e^{st}$

$$G(s) = \int_{-\infty}^{+\infty} g(\tau) e^{-s\tau} d\tau$$

$s = j\omega$

$e^{j\omega t}$ — $G(j\omega)e^{j\omega t}$

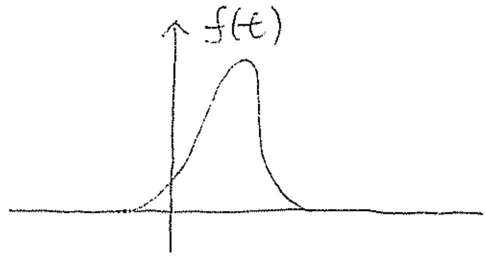
prosto periodična pobuda — prosto periodična funkcija

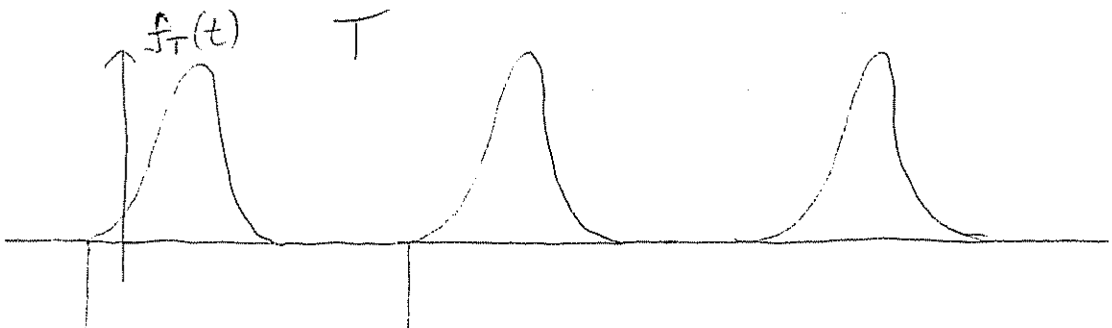
$$f(t) = \sum_{n=-\infty}^{+\infty} \underline{D}_n e^{jn\omega_0 t}$$

$$\sum_{n=-\infty}^{+\infty} \underbrace{\underline{D}_n G(jn\omega_0)}_{\underline{D}_n'} e^{jn\omega_0 t}$$

ovo je takođe složeno periodična funkcija
ista perioda kao pobuda
pobuda i odziv su isti

Predstavljanje aperiodičnih signala
Furijerova transformacija





ovo je složena periodična funkcija

1° $f_T(t) \rightarrow$ razviti u Furijerov red

2° $T \rightarrow \infty$ $\quad f_T(t) \rightarrow f(t)$

KORAK 1°

$$f_T(t) = \sum_{n=-\infty}^{+\infty} \underline{D}_n e^{jn\omega_0 t}$$

$$\underline{D}_n = \frac{1}{T} \int_{\underline{\ }} f_T(t) e^{-jn\omega_0 t} dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_0 t} dt$$

$$\underline{F}(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$$

$$T\underline{D}_n = \int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} dt$$

KORAK 2°

$T \to \infty$

$$\int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_0 t} dt \to \int_{-\infty}^{+\infty} f(t) e^{-jn\omega_0 t} = \underline{F}(n\omega_0)$$

$T \to \infty \qquad T\underline{D}_n \to F(n\omega_0)$

$$T\underline{D}_n \approx F(n\omega_0)$$

$$\underline{D}_n \approx \frac{F(n\omega_0)}{T}$$

$$f_T(t) \approx \sum_{n=-\infty}^{+\infty} \frac{F(n\omega_0)}{T} \cdot e^{jn\omega_0 t}$$

$$T = \frac{2\pi}{\omega_0}$$

$T \to \infty \qquad \omega_0 \to 0$

$$\omega_0 = \Delta\omega$$

$$f_T(t) \approx \sum_{n=-\infty}^{+\infty} \frac{\Delta\omega F(n\omega_0)}{2\pi} e^{jn\omega_0 t}$$

$T \to \infty$

$$f(t) = \lim_{\substack{T\to\infty \\ \omega_0=\Delta\omega\to 0}} \sum_{n=-\infty}^{+\infty} \frac{\Delta\omega F(n\omega_0)}{2\pi} e^{jn\omega_0 t}$$

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underline{F}(\omega) e^{j\omega t} d\omega}$$

složeno periodično

$$f(t) = \sum_{n=-\infty}^{+\infty} D_n e^{jn\omega_0 t} \qquad 0, \pm\omega_0, \pm 2\omega_0, \pm 3\omega_0$$

Zadnja relacija pokazuje da smo aperiodičan signal predstavljali kao sumu po svim mogućim vrijednostima ω, za razliku od složenoperiodičnog signala koji smo razvili u sumu po diskretnim vrijednostima ω.

$$\boxed{\underline{F}(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt}$$

↓

naziva se Furijerova transformacija signala ili funkcije aperiodično $f(t)$

$$\boxed{\underline{F}(\omega) = \mathcal{F}[f(t)]}$$

↓ Furijerova transformacija

Obratno ako je poznata Furijerova transformacija onda funkciju $f(t)$ možemo naći po formuli (*)

$$\boxed{f(t) = \mathcal{F}^{-1}[\underline{F}(\omega)]} \rightarrow \text{inverzna Furijerova transfor.}$$

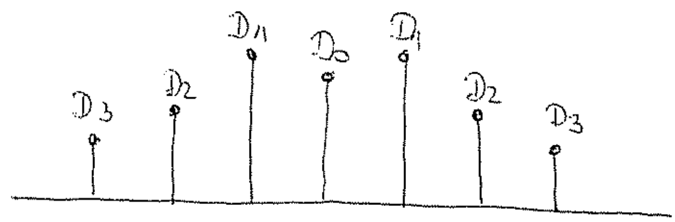
$$\boxed{f(t) \longleftrightarrow \underline{F}(\omega) \longrightarrow \mathcal{F} \quad \longleftarrow \mathcal{F}^{-1}}$$

Funkcija $\underline{F}(\omega)$ naziva se još i Furijeov spektar aperiodičnog signala $f(t)$

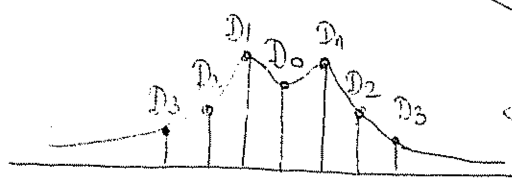
$\underline{F}(\omega) \longrightarrow \underline{D}_n$ složeni periodični signal

$\underline{F}(\omega)$ ↓ kontinualna funkcija

$\underline{D}_n$ ↓ diskretna veličina



$$\underline{D}_n \approx \frac{F_n(n\omega_0)}{T} \qquad \omega_0 = \frac{2\pi}{T}$$



oblik ove funkcije je isti! samo se smanjuje i postaje gušća

$\underline{D}_n \rightarrow \underline{F}(\omega)$ (ide u kontinualnu funkciju)

$|\underline{D}_n| = D_n$ amplitudni spektar

$\arg\{\underline{D}_n\} = \varphi_n$ fazni spektar

} obadvije su diskretne

$\underline{F}(\omega)$

$|\underline{F}(\omega)| = F(\omega)$ amplitudski spektar

$\arg \underline{F}(\omega)$ fazni spektar

Amplitudski spektar $F(\omega)$ je parna funkcija.

Fazni spektar je neparna funkcija.

VJEŽBE

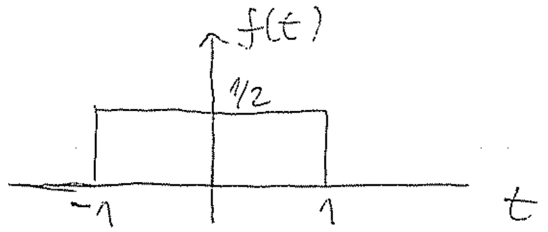
$$\sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

$$\cos x = \frac{e^{jx} + e^{-jx}}{2}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

① Odrediti $f(t)$ Furijerovu transformaciju signala sa slike.



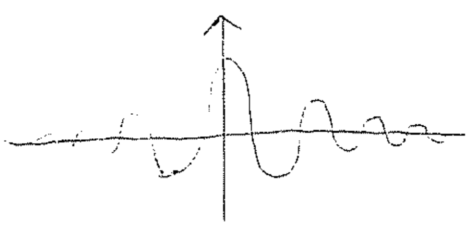
$$F(\omega) = \int_{-1}^{1} \frac{1}{2} e^{-j\omega t} dt$$

$$= \frac{1}{2}\left(-\frac{1}{j\omega}\right) e^{-j\omega} \Big|_{-1}^{1}$$

$$= \frac{e^{-j\omega} - e^{j\omega}}{-2j\omega}$$

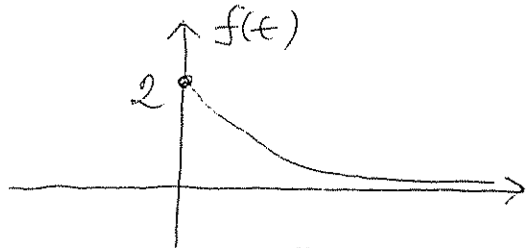
$$= \frac{\sin \omega}{\omega}$$

$\frac{\sin x}{x}$ $\Rightarrow$



$\frac{\sin \omega}{\omega}$

(2)



$$f(t) = 2e^{-2t} h(t) \quad \rightarrow \text{dato je}$$

Hevisajd znači nema ništa iza 0-le tj. sa lijeve strane

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} 2e^{-(2+j\omega)t} dt$$

$$= 2\int_{0}^{\infty} e^{-(2+j\omega)t} dt$$

$$= -\frac{2}{2+j\omega} \cdot e^{-(2+j\omega)} \Big|_{0}^{\infty}$$

$$= -\frac{2}{2+j\omega} \left[\cancel{e^{-\infty}}^{\,0} - \cancel{e^{0}}^{\,1} \right]$$

$$= \frac{-2}{2+j\omega} (0-1)$$

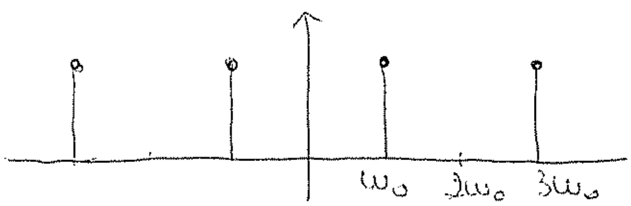
$$F(\omega) = \frac{2}{2+j\omega}$$

AMPLITUDSKI
↓

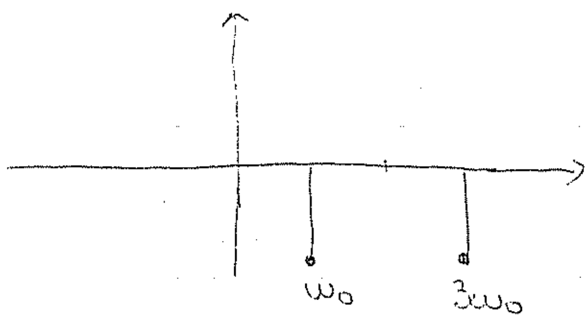
$$|F(\omega)| = \frac{2}{\sqrt{2^2+\omega^2}} = \frac{2}{\sqrt{4+\omega^2}}$$

FAZNI
↓

$$\theta(\omega) = \angle 2 - \angle 2+j\omega$$
$$= 0 - \text{arctg}\,\frac{\omega}{2}$$
$$= -\text{arctg}\,\frac{\omega}{2}$$

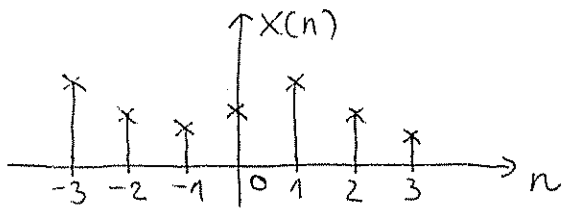


$$F(n) = |F_n|\, e^{j\theta\omega}$$



II KOLOKVIJUM

DISKRETNI SIGNALI

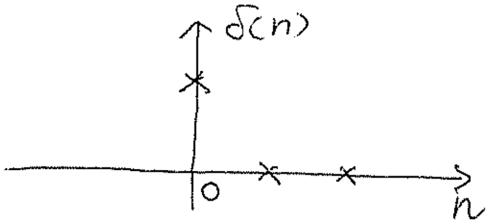


x(n) definisan samo za cjelobrojno n

PRIMJERI DISKRETNIH SIGNALA

1° JEDINIČNI SIGNALI

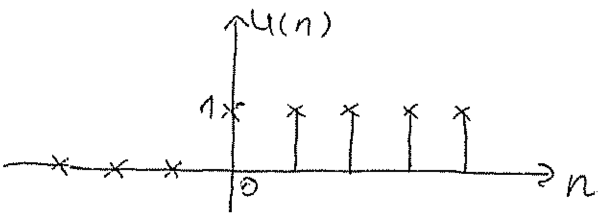
$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



⇒ ovo je realan signal i upotrebljava se

2° JEDINIČNI STEP NIZ

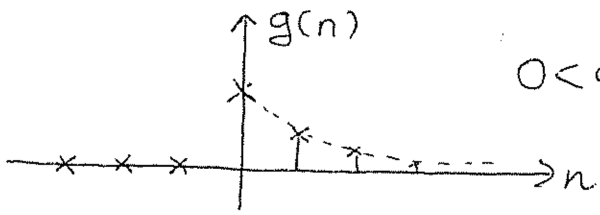
$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



$$\boxed{u(n) = \sum_{k=-\infty}^{\infty} \delta(k)}$$

3° REALNI EXP SIGNAL (NIZ)

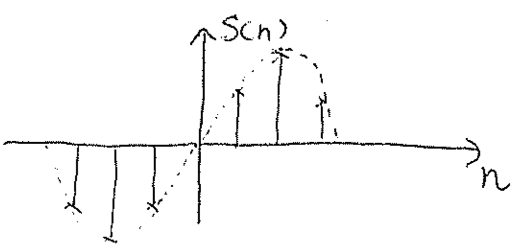
$$g(n) = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



$0 < a < 1$

4° SINUSNI NIZ

$$S(n) = A \cdot \sin(\omega_0 n + \Theta)$$



PERIODIČNI NIZ

$x(n)$ je periodičan ako je:

$$x(n+N) = x(n)$$

gdje je N - perioda i mora biti cijeli broj $(\in Z)$

PRIMJER

Ispitati periodičnost i ispitati periodu signala.

a) $x(n) = e^{j\frac{2\pi n}{3}}$

b) $x(n) = \cos \frac{2\pi}{6/7} \cdot n$

c) $x(n) = \sin 3n$

a) $x(n+N) = e^{j(n+N)} = e^{j\left(\frac{2\pi n}{3} + \frac{2\pi N}{3}\right)} =$

$$\underbrace{e^{j\frac{2\pi n}{3}}}_{x(n)} \cdot e^{j\frac{2\pi N}{3}} = x(n) \cdot e^{j\frac{2\pi N}{3}}$$

$$x(n+N) = x(n) \Rightarrow e^{j\frac{2\pi N}{3}} = 1$$

$$\cos \frac{2\pi N}{3} = 1 \Rightarrow \frac{\cancel{2}\cancel{\pi} N}{3} = \cancel{2}K\cancel{\pi} = N = 3K$$

$$K = 1 \Rightarrow \boxed{N=3}$$

niz je periodičan sa osnovnom periodom $N=3$

b) $x(n+N) = \cos \frac{2\pi}{6/7}(n+N) =$

$$\cos\left(\frac{2\pi}{6/7} n + \frac{2\pi}{6/7} N\right) \underset{\uparrow}{=} \cos \frac{2\pi}{6/7} n$$

Treba da bude

$$\frac{\cancel{2\pi}}{6/7} N = \cancel{2}K\cancel{\pi} \quad | 2\pi$$

$$N = \frac{6}{7} K$$

$K = 7 \Rightarrow \boxed{N=6}$ periodičan je.

c) $x(n+N) = \sin 3(n+N) = \sin(3n+3N) \underset{\uparrow}{=} \sin 3n$

Treba da bude

$3N = 2k\pi$

$N = \frac{2k\pi}{3} \notin Z$

signal nije periodičan

$x(n)$ može da se zapiše u obliku sume jediničnih elemenata.

$$\boxed{x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)}$$

Dokaz

$$\delta(n-k) = \begin{cases} 1, & n-k=0 \Rightarrow k=n \\ 0, & n-k\neq 0 \Rightarrow n\neq k \end{cases}$$

$$\sum_{k=-\infty}^{\infty} x(k)\delta(n-k) \underset{\uparrow}{=} x(n)\cdot 1 = x(n)$$

postoji samo za $k=n$

Parametri za opisivanje diskretnih signala

1° Amplituda signala

Maksimalna apsolutna vrijednost signala

$$\boxed{M = \max_{-\infty<n<+\infty} |x(n)|}$$

2° Energija signala

$$\boxed{E = \sum_{n=-\infty}^{\infty} |x(n)|^2}$$

3° SNAGA SIGNALA

$$\boxed{P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2}$$

LINEARNI VREMENSKI INVARIJANTNI SISTEM (LVIS)

LVIS skraćenica LTIS

$$\boxed{y(n) = T\,[x(n)]}$$

izlazni signal — operator algoritma — ulazni signal

$T[\;]$ ⇒ transformacija kojom sistem od ulaznog signala daje izlazni signal

T ⇒ određuje sistem

1° Sistem je linearan kada je T linearno:

$$T\,[Ax_1(n)+Bx_2(n)] = AT\,[x_1(n)] + BT\,[x_2(n)]$$

za ovaj sistem se kaže da je linearan

2° Sistem je vremenski invarijantan ako je:

$$y(n) = T\,[x(n)] \Rightarrow y(n+N) = T\,[x(n-N)]$$

Primjer

Ispitati linearnost sledećih sistema:

a) $y(n) = T\,[x(n)] = x^2(n)$

b) $y(n) = T\,[x(n)] = x(n) - x(n-1)$

a) $T\,[Ax_1(n)+Bx_2(n)] = (Ax_1(n)+Bx_2(n))^2 =$

$$= A^2 x_1^2(n) + 2AB x_1(n) x_2(n) + B^2 x_2^2(n)$$

$$AT[x_1(n)] + BT[x_2(n)] = A x_1^2(n) + B x_2^2(n)$$

sistem nije linearan

b) $T[Ax_1(n) + Bx_2(n)] =$

$Ax_1(n) + Bx_2(n) - (Ax_1(n-1) + Bx_2(n-1)) =$

$A(x_1(n) - x_1(n-1)) + B(x_2(n) - x_2(n-1)) =$

$AT[x_1(n)] + BT[x_2(n)]$

sistem je linearan

PRIMJER

Ispitati vremensku invarijantnost

a) $y(n) = T[x(n)] = x(n) + x(n-1)$

b) $y(n) = T[x(n)] = n x(n)$

a) $T[x(n-N)] = x(n-N) + x(n-N-1) = y(n-N)$

sistem je vremenski invarijantan

b) $T[x(n-N) = n x(n-N)$

$y(n-N) = (n-N) x(n-N)$

sistem nije vremenski invarijantan

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

$\Downarrow$

$$\underset{\substack{\downarrow \\ \text{za linearan sistem}}}{y(n)} = T[x(n)] = T\left[\sum_{-\infty}^{\infty} x(k)\delta(n-k)\right] = \sum_{k=-\infty}^{\infty} x(k) T[\delta(n-k)] \underset{\substack{\uparrow \\ \text{vremenski invarijant.}}}{=}$$

$$= \sum_{k=-\infty}^{\infty} x(n) h(n-k) \Rightarrow \boxed{x(n) * h(n) = h(n) * x(n)}$$

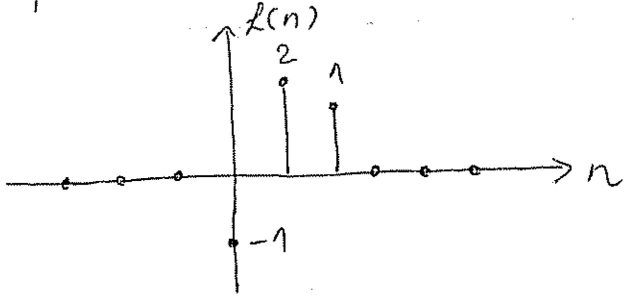
konvolucibna suma

$h(n) = T[\delta(n)]$

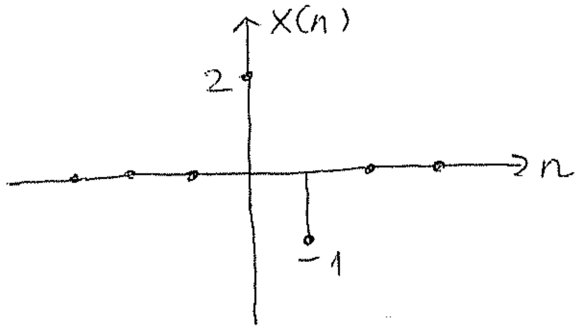
$\downarrow$ jedinični odziv

Razlike između konvolucionog integrala i konvolucione sume

Impulsni odziv sistema je dat na slici:

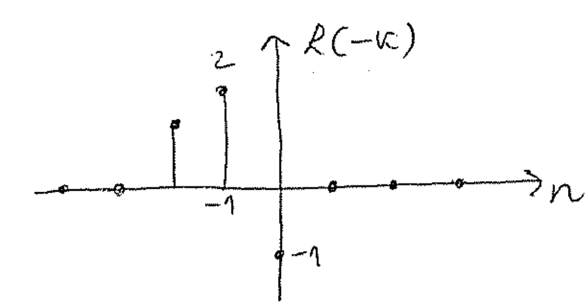


Naći odziv sistema na ulazni niz x(n) predstavljen na slici

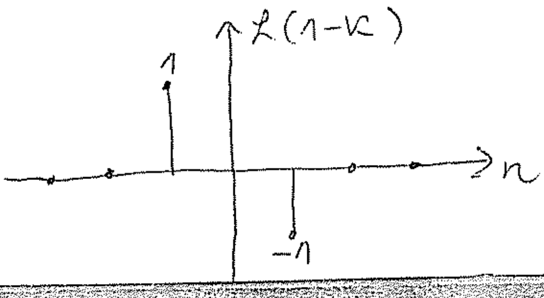


$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

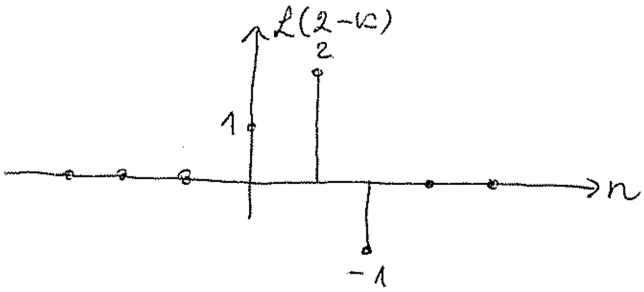
$$n=0,\ y(0) = \sum_{k=-\infty}^{\infty} x(k)\,h(-k) = 2\cdot(-1) = -2$$ gledamo i množimo grafike x(n) i h(−k)



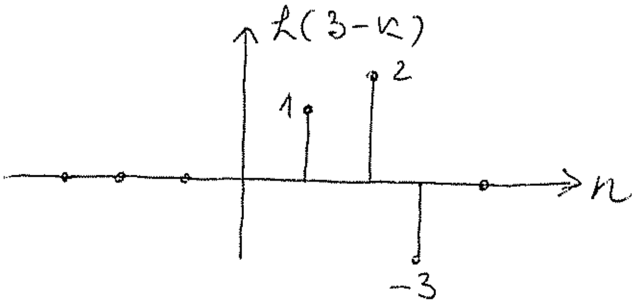
$$n=1,\ y(1) = \sum_{k=-\infty}^{\infty} x(k)\,h(1-k) = 2\cdot 2 + (-1)\cdot(-1) = 4+1 = 5$$ x(n) i h(1−k)



$n=2$, $y(2)=\sum_{k=-\infty}^{\infty} x(k)\,h(2-k)=2\cdot 1+(-1)\cdot 2=2-2=0$ $\quad x(n)$ i $h(2-k)$

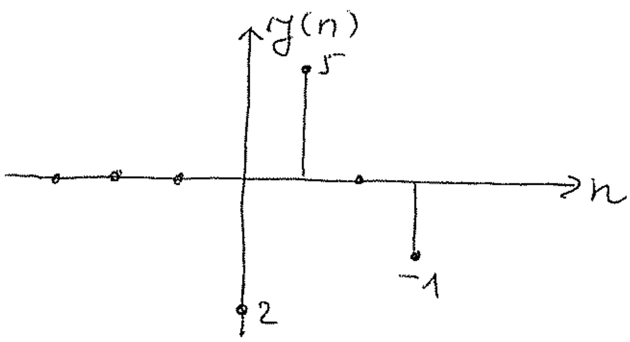


$n=3$, $y(3)=\sum_{k=-\infty}^{\infty} x(k)\,h(3-k)=(-1)\cdot 1=-1$ $\quad x(n)$ i $h(3-k)$



$y(n)=0 \quad n\geq 4$

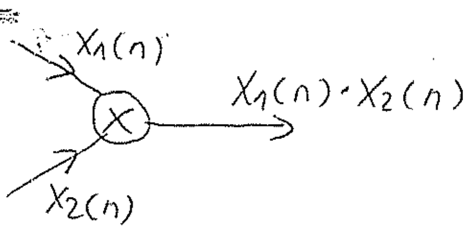
$y(4)=0 \quad n<0$

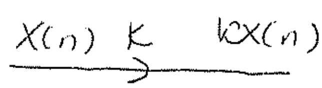


Realizacija sistema može biti : SOFTVERSKA

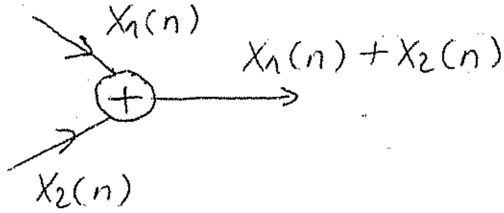
HARDVERSKA

- MNOŽAČ

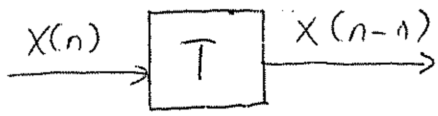




- sabirač



- kolo za kašnjenje



- kauzalnost i stabilnost

Kauzalan sistem – ne postoji odziv prije pobude

$h(n)=0 \quad , \quad n<0$

$x(n)=0 \quad , \quad n<0$ ⇒ kaže se da je kauzalan niz

Stabilnost – ako je $|x(n)|<M, \forall n$ tada je da bi sistem bio stabilan:

$$|y(x)|<\infty$$

Dokaz

$$|y(n)|=\left|\sum_{k=-\infty}^{\infty}x(k)h(n-k)\right|\le\sum_{k=-\infty}^{\infty}\underbrace{|x(k)|}_{M}|h(n-k)|<$$

$$<M\boxed{\sum_{k=-\infty}^{\infty}|h(n-k)|}<\infty$$

$$\sum_{k=-\infty}^{\infty}|h(k)|<\infty$$

VJEŽBE $u(-n)$ $-1\,u(5-n)$ $5-n=0$, $-n=-5$, $n=5$ $\delta(n+1)$ $n=-1$

① Dati su diskretni signali:

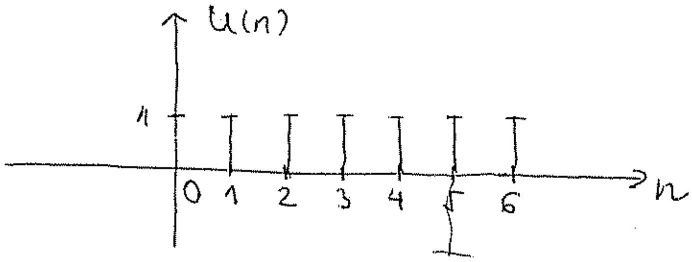
$$x(n) = u(n) - u(n-5) - \delta(n-2)$$

$$y(n) = \delta(n) + 2\delta(n-1) + \delta(n-2)$$

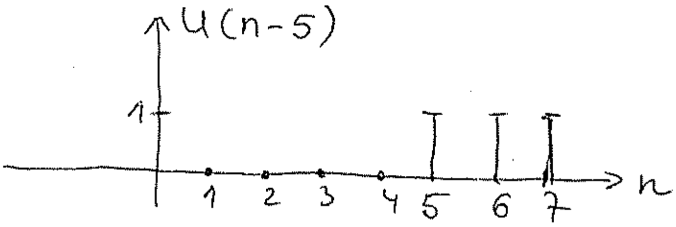
a) predstaviti ih grafički

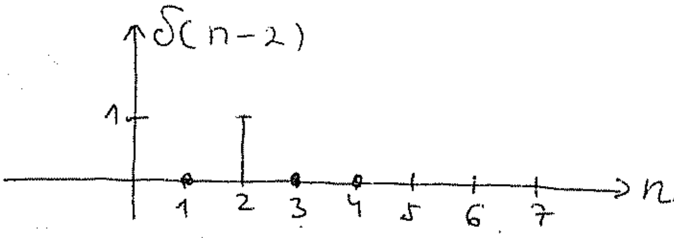
b) odrediti konvoluciju $z(n) = x(n) * y(n)$

a)

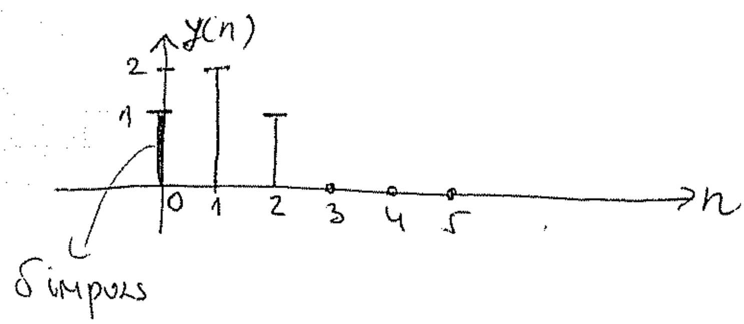


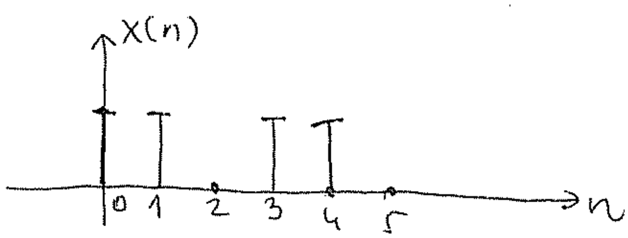
$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & \text{ostalo } n \end{cases}$$





$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & \text{za ostalo} \end{cases}$$



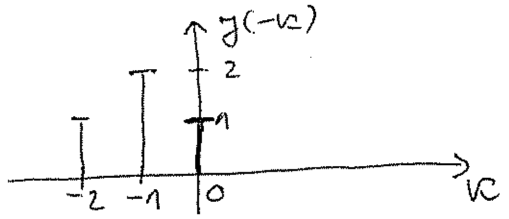


b) $z(n) = x(n) * y(n)$

$$\sum_{k=-\infty}^{\infty} x(k)\, y(n-k)$$

za $n=0$

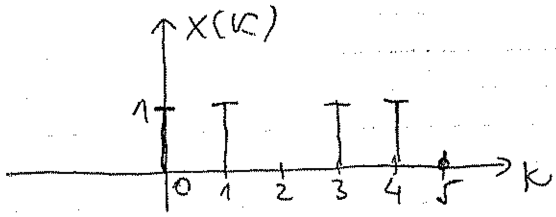
$$z(0) = \sum_{k=-\infty}^{\infty} x(k)\, y(-k) = 1\cdot 1 = 1$$



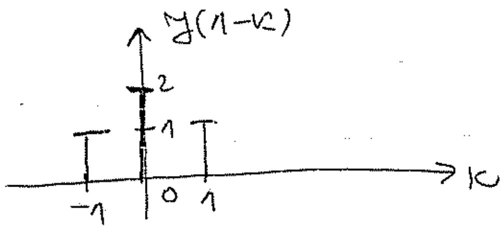
SIMETRIČAN U ODNOSU NA y OSU ($y(n)$)

za $n=1$

$$z(1) = \sum_{k=-\infty}^{\infty} x(k)\, y(1-k) = 1\cdot 1 + 2\cdot 1 = 3$$

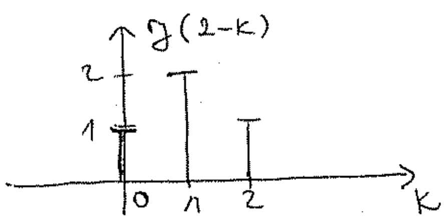


⇐ sve tr množimo sa $x(k)$



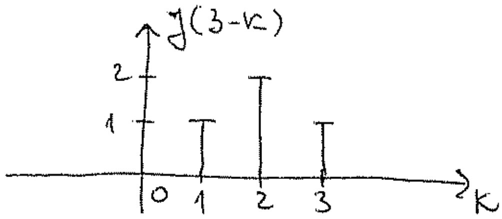
za $n=2$

$$z(2) = \sum_{k=-\infty}^{\infty} x(k)\, y(2-k) = 1\cdot 1 + 2\cdot 1 = 3$$



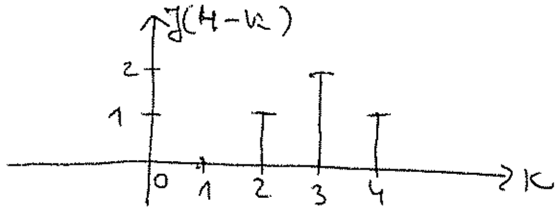
za $n=3$

$$z(3)=\sum_{k=-\infty}^{\infty} x(k)\, y(3-k) = 1\cdot 1 + 1\cdot 1 = 2$$



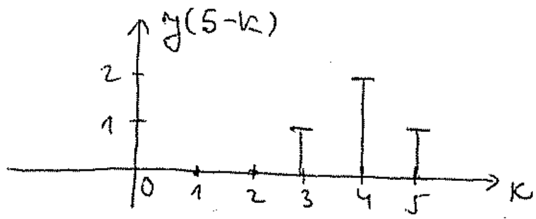
za $n=4$

$$z(4)=\sum_{k=-\infty}^{\infty} x(k)\, y(4-k) = 2\cdot 1 + 1\cdot 1 = 3$$



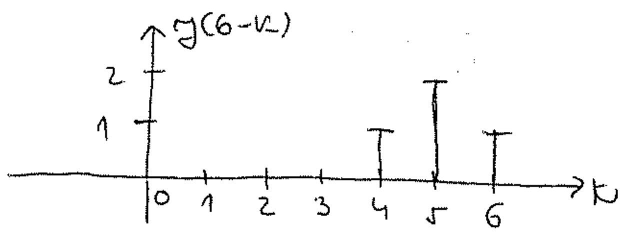
za $n=5$

$$z(5)=\sum_{k=-\infty}^{\infty} x(k)\, y(5-k) = 1\cdot 1 + 2\cdot 1 = 3$$



za $n=6$

$$z(6)=\sum_{k=-\infty}^{\infty} x(k)\, y(6-k) = 1\cdot 1 = 1$$



za $n=7$

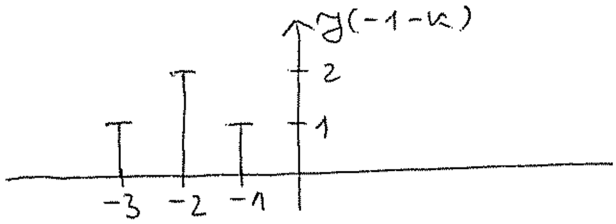
$$z(7)=\sum_{k=-\infty}^{\infty} x(k)\cdot y(7-k)$$

$z(7) = 0$

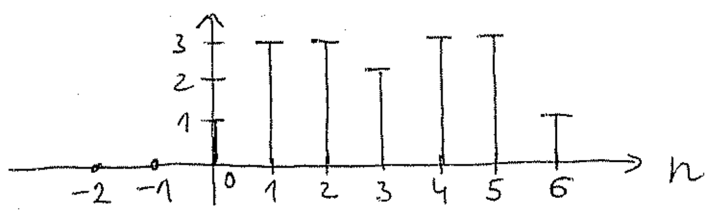
$z(n) = 0$ za $n > 6$

(jer množimo sa grafikom $x(k)$ sa 5-com a to je 0-la)

$$z(-1) = \sum_{k=-\infty}^{\infty} x(k) y(-1-k)$$

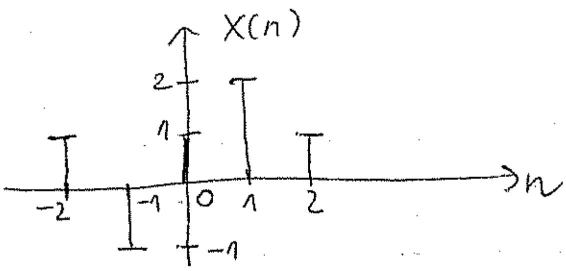


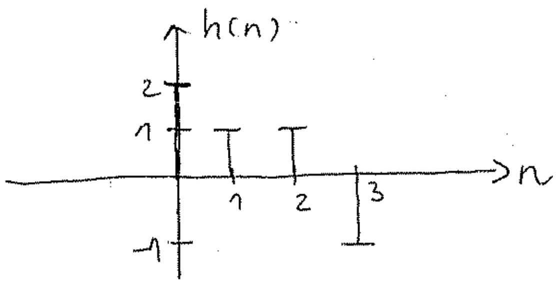
$z(n) = 0$ za $n < 0$



② zadatak za vježbu

Odrediti konvoluciju signala datu na slici:





PREDAVANJE

DIFERENCNE JEDNAČINE

- ovim jednačinama se vežu ulazni i izlazni signali.

$$\sum_{n=0}^{M} B_i\, x(n-1) = \sum_{y=0}^{N} A_j\, y(n-j)$$

Primjer

Data je diferencna jednačina prvog reda

$$y(n) = -A_1 y(n-1) + B_0 x(n) + B_1 x(n-1)$$

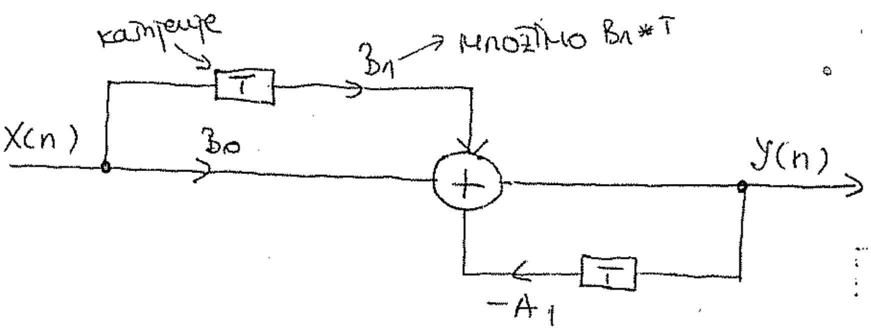
pretpostavljajući da je sistem kauzalan odrediti njegov impulsni odziv $h(n) = ?$

NAPOMENA:

Realizacija na 2 načina:

1° SOFTVERSKI (računski)

2° HARDVERSKI



$\Rightarrow$ HARDVERSKI SISTEM

$X(n) = \delta(n)$, $y(n) = h(n) = ?$

Kada je sistem kauzalan slijedi:

$h(n) = 0$, $n < 0$

$$h(0) = -A_1 \cdot \cancel{h(-1)} + B_0\delta(0) + B_1\cancel{\delta(-1)} = B_0$$

$$h(1) = -A_1 \cdot h(0) + B_0\cancel{\delta(1)} + B_1\delta(0)$$
$$= -A_1 B_0 + B_1$$

$$h(2) = -A_1 \cdot h(1) + \cancel{B_0\delta(2)} + \cancel{B_1\delta(1)}$$
$$= -A_1(-A_1 \cdot B_0 + B_1)$$

$$h(3) = -A_1 \cdot h(2) + B_0\cancel{\delta(3)} + \cancel{B_1\delta(2)}$$
$$= (-A_1)^2 \cdot (-A_1 B_0 + B_1)$$

$$\underline{h(n) = (-A_1)^{n-1} \cdot (-A_1 \cdot B_0 + B_1)} \quad , \quad n \geq 1$$

$h(n) = B_0\delta(n)$ → Da bi bila samo u nuli dodajemo $\delta(n)$

$$h(n) = B_0\delta(n) + (-A_1)^{n-1}(-A_1 B_0 + B_1)u(n-1)$$

NAPOMENA:

IIR – sistemi sa beskonačnim impulsnim odzivom

FIR – sistemi sa konačnim impulsnim odzivom

FURIJEROVA TRANSFORMACIJA DISKRETNIH SIGNALA

$$\boxed{X(e^{j\omega}) = FT[x(n)] \overset{Def.}{=} \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}}$$

↓

Furijerova transformacija diskretnog signala (proizvoljnog signala $x(n)$)

$$\boxed{H(e^{j\omega}) = FT[h(n)] = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n}}$$

↓

Furijerova transformacija impulsnog odziva i naziva se FREKVENTNI ODZIV SISTEMA

$$e^{-j(\omega+2\pi)n} = e^{-j\omega n} \cdot \underbrace{e^{-j2\pi n}}_{\underbrace{\cos 2\pi n}_{1} - j\underbrace{\sin 2\pi n}_{0}} = e^{-j\omega n}$$

$$\Downarrow$$

$X(e^{j\omega})$ → periodična funkcija po frekventnoj (ω) osi perioda 2π

Inverzna FT

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega$$

Primjer:

Odrediti Furijerovu transformaciju diskretnog signala

$x(n) = a^n u(n) \quad , \quad |a| < 1$

$FT[x(n)] = ?$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = \sum_{n=-\infty}^{\infty} a^n u(n) e^{-j\omega n} =$$

$$= \sum_{n=0}^{+\infty} a^n e^{-j\omega n} = \sum_{n=0}^{+\infty} (a \cdot e^{-j\omega})^n$$

ovo je suma geometrijskog reda

$$|a \cdot e^{-j\omega}| = |a| \cdot |e^{-j\omega}| = |a| \cdot 1 = |a| < 1$$

$$\Downarrow$$

geometrijski red sa članom $|a| < 1$

$$|e^{jx}| = |\cos x + j \sin x|$$
$$= \sqrt{\cos^2 x + \sin^2 x}$$
$$= \sqrt{1} = 1$$

za geometrijski red važi:

$$\sum_{n=0}^{N-1} x^n = \frac{1-x^N}{1-x}$$

ako je $\left.\begin{matrix}|x|<1 \\ N\to\infty\end{matrix}\right\} \Rightarrow x^N \to 0$

onda je $\sum_{n=0}^{N-1} x^n = \frac{1}{1-x}$

$$X(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}} = \frac{1}{1-a\cos\omega + aj\sin\omega}$$

* AMPLITUDSKA KARAKTERISTIKA

$$|X(e^{j\omega})| = \frac{1}{\sqrt{(1-a\cos\omega)^2+(a\sin\omega)^2}}$$

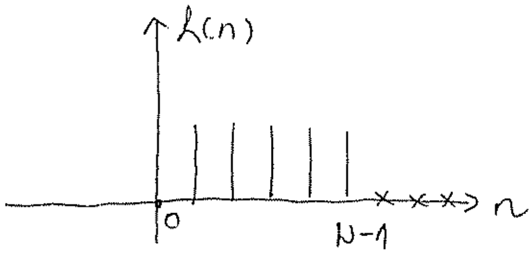
* FAZNA KARAKTERISTIKA

$$\arg(X(e^{j\omega})) = \arg(1) - \arg(1-a\cos\omega + ja\sin\omega)$$
$$= -\operatorname{arctg}\frac{a\sin\omega}{1-a\cos\omega}$$

PRIMJER:

Odrediti Furijerovu transformaciju signala

$h(n) = u(n) - u(n-N)$



$\Rightarrow$ ovo je pravougaoni prozor

od 0 do N-1 su 1-ce

od N-1 pa nadalje su 0-le

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n} = \sum_{n=0}^{N-1} e^{-j\omega n}$$

$$= \frac{1-(e^{-j\omega})^N}{1-e^{-j\omega}} = \frac{1-e^{-j\omega N}}{1-e^{-j\omega}}$$

$$= \frac{e^{-j\omega\frac{N}{2}}}{e^{-j\frac{\omega}{2}}} \cdot \frac{e^{j\omega\frac{N}{2}} - e^{-j\omega\frac{N}{2}}}{e^{j\frac{\omega}{2}} - e^{j\frac{\omega}{2}}}$$

$$\cos\frac{\omega}{2} + j\sin\frac{\omega}{2} - \left(\cos\frac{\omega}{2} - j\sin\frac{\omega}{2}\right)$$

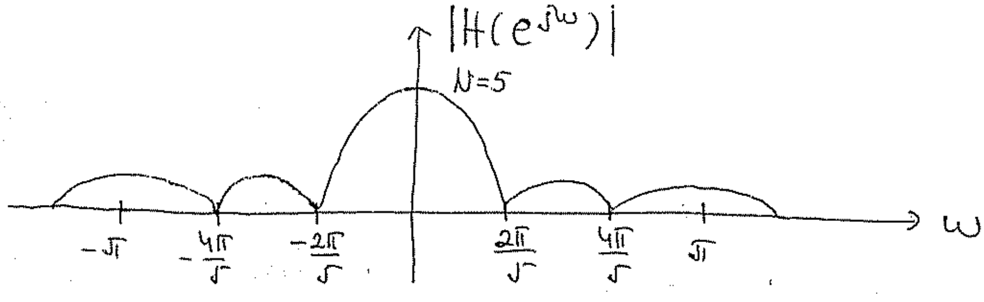
$$= e^{-j\frac{\omega}{2}(N-1)} \cdot \frac{2j\sin\frac{\omega N}{2}}{2j\sin\frac{\omega}{2}}$$

$$= e^{-j\omega\frac{N-1}{2}} \cdot \frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}} = H(e^{j\omega})$$

Apsolutna vrijednost

$$|H(e^{j\omega})| = \left| \frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}} \right|$$

za $N=5$



$$\sin\frac{\omega N}{2} = 0$$

$$\frac{\omega N}{2} = k\pi$$

$$\omega = \frac{2k\pi}{N} = \frac{2k\pi}{5}$$

u 0-li će imati vrijednost 5

Osobine FT diskretnih signala

1° **Linearnost**

$$FT[ax(n)+by(n)] = aX(e^{j\omega})+bY(e^{j\omega})$$

2° **Pomjeranje po vremenu**

$$FT[x(n-n_0)] = e^{-j\omega n_0}X(e^{j\omega})$$

3° **Modulacija**

$$FT[x(n)\cdot e^{j\omega_0 n}] = X(e^{j(\omega-\omega_0)})$$

4° **Konvolucija signala**

$$FT[x(n)*h(n)] = X(e^{j\omega})\cdot H(e^{j\omega}) = Y(e^{j\omega})$$

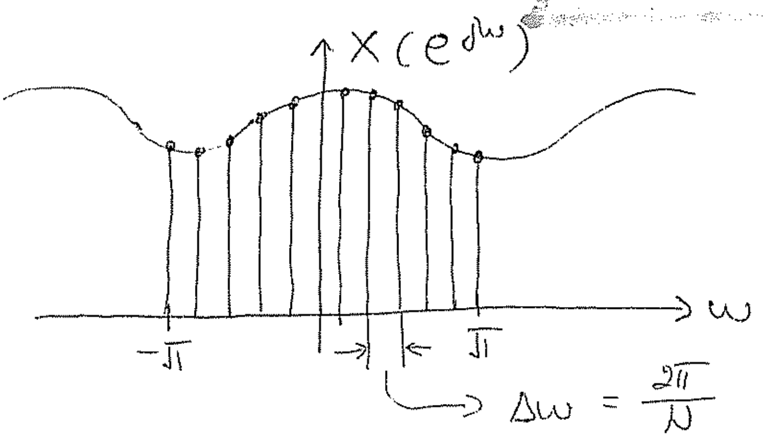
5° **Proizvod nizova**

$$FT[x(n)h(n)] = X(e^{j\omega}) * H(e^{j\omega})$$

Napomena:

$$\boxed{\sum_{n=-\infty}^{\infty} e^{-j\omega n} = 2\pi\,\delta(\omega)}$$

Diskretna FT (DFT)



$$\boxed{\Delta\omega = \frac{2\pi}{N}} \rightarrow \text{korak odabiranja}$$

$N = ?$

$$X(e^{j\omega}) = FT[x(n)]$$

odabiranjem

$$X_p(k) - DFT[x_p(n)]$$

perioda je N

$N \geq$ trajanje (dužina) signala $x(n)$

Definicija DFT

$$DFT[x_p(n)] = X_p(k) = \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi}{N}nk}$$

$x_p(n) = x(n)$, za $0 \leq n \leq N-1$

$e^{-j\frac{2\pi}{N}}$ - označava se sa W_N

$$\boxed{X_p(k) = \sum_{n=0}^{N-1} x_p(n) W_N^{nk}} \rightarrow \text{Definicija DFT}$$

Definicija IDFT (inverzna)

$$\boxed{x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_p(k) W_N^{-nk}} \rightarrow \text{definicija IDFT}$$

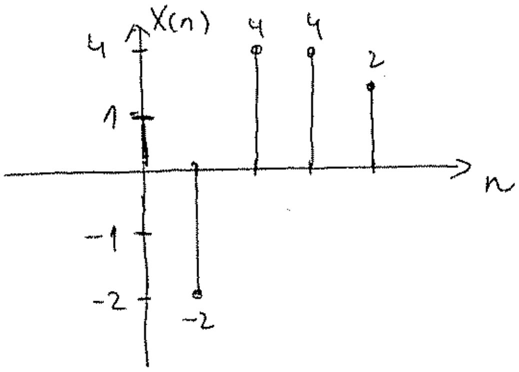
NAPOMENA:

Periodični diskretni signal ne može imati svoju Furijerovu transformaciju jer ne može konvergirati, a može imati samo diskretnu Furijerovu transformaciju.

Primjeri:

Odrediti DFT signal.

$$x(n)=\delta(n)-2\delta(n-1)+4\delta(n-2)+4\delta(n-3)+2\delta(n-4)$$



Moramo uzeti $N \geq 5$ tj. $N=5$

$$X_p(k)=\sum_{n=0}^{4} x(n)\cdot e^{-j\frac{2\pi}{5}nk}$$

$$=\sum_{n=0}^{4}\left(\delta(n)-2\delta(n-1)+4\delta(n-2)+4\delta(n-3)+2\delta(n-4)\right)\cdot e^{-j\frac{2\pi}{5}nk}$$

$$=1-2e^{-j\frac{2\pi}{5}k}+4e^{-j\frac{2\pi}{5}2k}+4e^{-j\frac{2\pi}{5}3k}+2e^{-j\frac{2\pi}{5}4k}$$

$$=1-2e^{-j\frac{2\pi}{5}k}+2e^{j\frac{2\pi}{5}k}+4e^{-j\frac{2\pi}{5}2k}+4e^{j\frac{2\pi}{5}2k}$$

$$=1+4j\sin\frac{2\pi}{5}k+8\cos\frac{2\pi}{5}2k$$

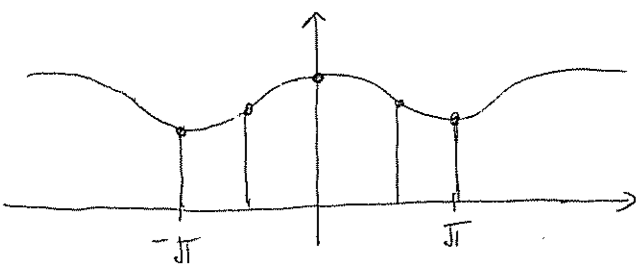
$$e^{-j\frac{2\pi}{5}\cdot 2(k+k_p)}$$

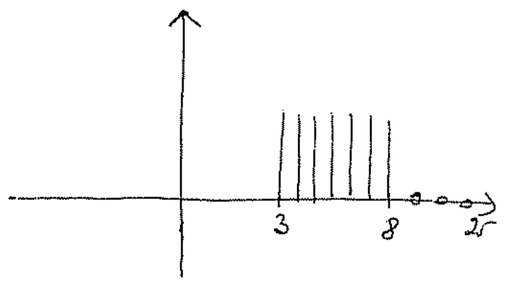
$$\frac{2\pi}{5}\cdot 2\cdot k_p=2l\pi$$

$$k_p=\frac{5l}{2},\quad l=2$$

$$k_p=5$$

Napomena:





signal traje od 3 do 8 ; dodajemo nule i sada traje od 3 do 25 i nije mu se izmijenila vrijednost i to se zove ZEROPEDDILNG

VJEŽBE

① Odrediti impulsni odziv kauzalnog sistema opisan jednačih
$y(n) - \frac{1}{2}y(n-1) = x(n) + 2x(n-1) + x(n-2)$
i provjeriti da li je sistem stabilan

$x(n) = \delta(n)$
$y(n) = h(n)$

$h(n) = 0$, za $n<0$

$h(n) = \frac{1}{2}h(n-1) + \delta(n) + 2\delta(n-1) + \delta(n-2)$

$h(0) = \frac{1}{2}\cancel{h(-1)}^{0} + \delta(0) + 2\cancel{\delta(-1)}^{0} + \cancel{\delta(-2)}^{0} = \delta(0) = 1$

$h(1) = \frac{1}{2}h(0) + \cancel{\delta(1)}^{0} + 2\delta(0) + \cancel{\delta(-1)}^{0} = \frac{1}{2}\cdot 1 + 2 = \frac{5}{2}$

$h(2) = \frac{1}{2}h(1) + \cancel{\delta(2)}^{0} + 2\cancel{\delta(1)}^{0} + \delta(0) = \frac{1}{2}\cdot\frac{5}{2} + 1 = \frac{9}{4}$

$h(3) = \frac{1}{2}h(2) + \cancel{\delta(3)}^{0} + 2\cancel{\delta(2)}^{0} + \cancel{\delta(1)}^{0} = \frac{1}{2}h(2) = \frac{1}{2}\cdot\frac{9}{4}$

$h(4) = \frac{1}{2}h(3)$ + ostalo su nule $= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{9}{4} = \frac{1}{2^2}\cdot\frac{9}{4}$

$h(5) = \frac{1}{2}h(4) = \frac{1}{2}\cdot\frac{1}{2^2}\cdot\frac{9}{4} = \frac{1}{2^3}\cdot\frac{9}{4}$

$h(n) = \left(\frac{1}{2}\right)^{n-2} \cdot \frac{9}{4}$, za $n \geq 2$

$h(0) = 1$

$h(1) = \frac{5}{2}$

provjera stabilnosti

stabilan je ako je $\sum_{n=-\infty}^{+\infty} |h(n)|$ suma konačna (konačan broj), a ako nije nego teži beskonačnosti onda nije stabilan.

$$1 + \frac{5}{2} + \frac{9}{4} + \frac{1}{2} \cdot \frac{9}{4} + \frac{1}{2^2} \cdot \frac{9}{4} + \frac{1}{2^3} \cdot \frac{9}{4} + \dots =$$

$$1 + \frac{5}{2} + \frac{9}{4}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots\right) = \text{(opadajući)}$$

$$1 + \frac{5}{2} + \frac{9}{4} \cdot \left(\frac{1}{1-\frac{1}{2}}\right) = 8$$

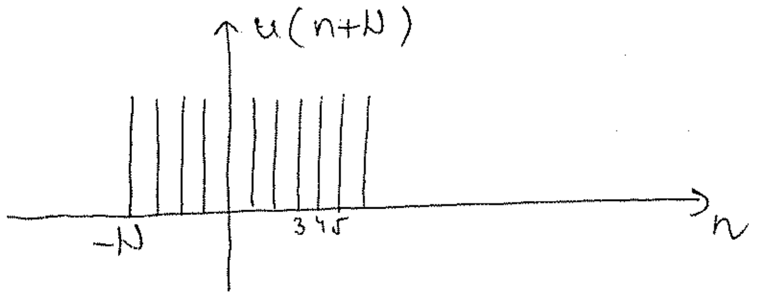
Naš sistem je stabilan

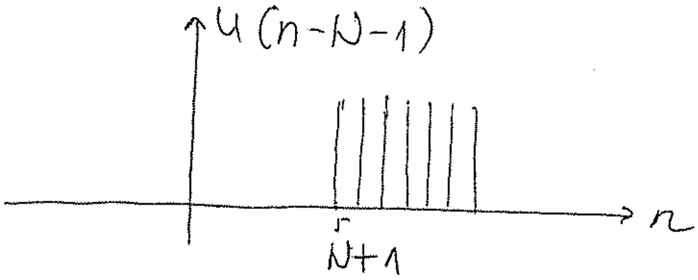
$|q| < 1$

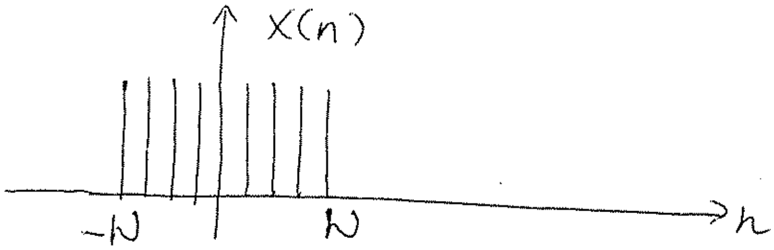
$\boxed{S = \frac{a}{1-q}}$ → suma beskonačnog geometrijskog reda

(2) Odrediti Furijerovu transformaciju signala

$x(n) = u(n+N) - u(n-N-1)$







$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=-N}^{N} 1 e^{-j\omega n}$$

$$= e^{j\omega N} \cdot \frac{1 - e^{j\omega(2N+1)}}{1 - e^{j\omega}}$$

$$= \frac{e^{j\omega N} \cdot e^{j\omega\left(\frac{2N+1}{2}\right)} \left(e^{-j\omega\frac{(2N+1)}{2}} - e^{j\omega\left(\frac{2N+1}{2}\right)}\right)}{e^{j\frac{\omega}{2}} \cdot \left(e^{-j\frac{\omega}{2}} - e^{j\frac{\omega}{2}}\right)}$$

$$= \frac{e^{j\omega N} \cdot e^{j\omega\left(\frac{2N+1}{2}\right)}}{e^{j\frac{\omega}{2}}} \cdot \frac{\sin\left(\omega\frac{(2N+1)}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

$$= \frac{e^{j\omega N} \cdot e^{j\omega N} \cdot e^{j\frac{\omega}{2}}}{e^{j\frac{\omega}{2}}} \cdot \frac{\sin\left(\omega\frac{(2N+1)}{2}\right)}{\sin\frac{\omega}{2}}$$

$$\boxed{\begin{aligned} \sin x &= \frac{e^{jx} - e^{-jx}}{2j} \\ \cos x &= \frac{e^{jx} + e^{-jx}}{2} \end{aligned}}$$

$n = -N$

$e^{j\omega N}$

$n = -N+1$

$e^{-j\omega(-N-1)} = e^{+j\omega N - j\omega} = \frac{e^{j\omega N}}{e^{j\omega}}$

$$e^{j\omega N} + \frac{e^{j\omega N}}{e^{j\omega}} + \frac{e^{j\omega N}}{e^{j2\omega}}$$

konačna progresija (njena suma)

$$\boxed{a\,\frac{1-q^n}{1-q}}$$

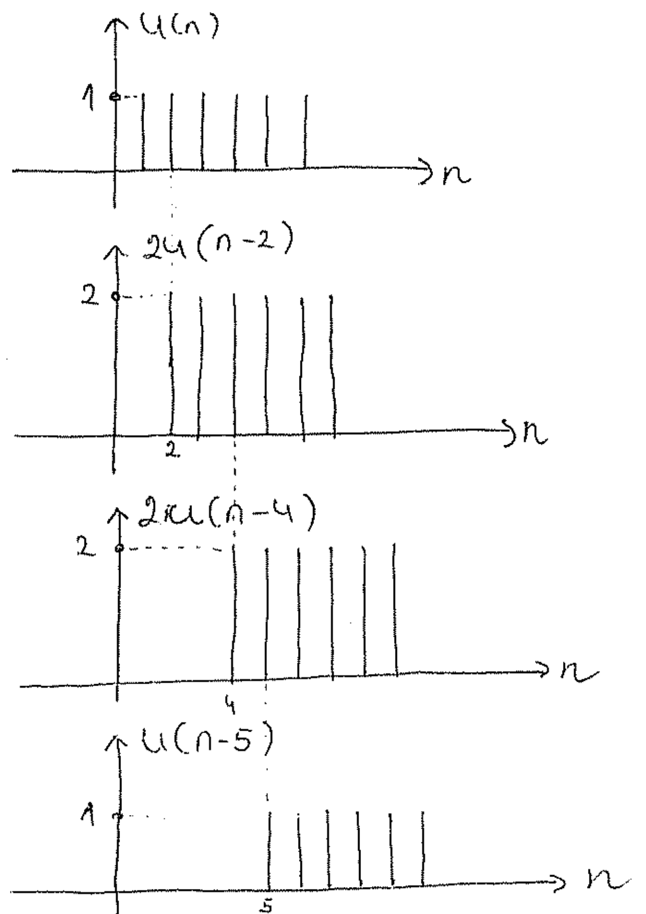
n - broj članova

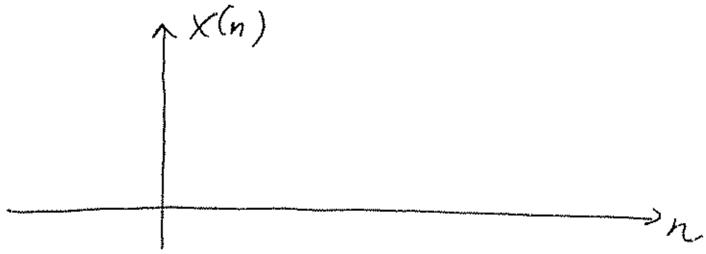
q - količnik

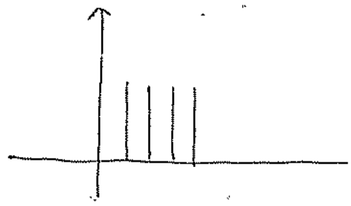
a - prvi član

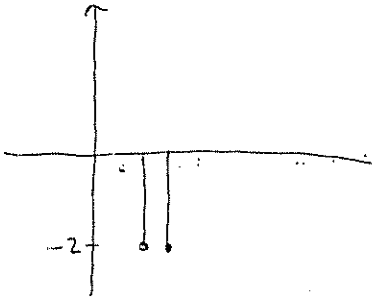
(3) Odrediti DFT niza

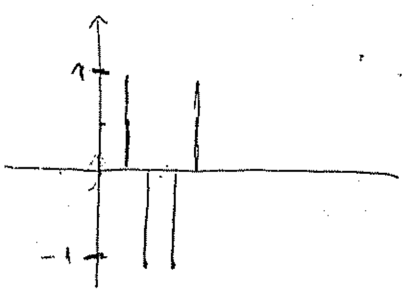
$x(n) = u(n) - 2u(n-2) + 2u(n-4) - u(n-5)$











$$X_p(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N} n \cdot k}$$

Dužina signala je 5

$$N \geq 5$$

uzimamo da je $N=5$

$$X_p(k) = \sum_{n=0}^{4} x(n)\, e^{-j\frac{2\pi}{5} nk}$$

$$= \underset{\substack{\downarrow \\ \text{za } n=0 \text{ tj. za } x(0)=1}}{1} + \underset{\downarrow n=1}{1} \cdot e^{-j\frac{2\pi}{5} nk} + \underset{\downarrow \text{za } n=2}{(-1)} \cdot e^{-j\frac{2\pi}{5} 2k} + \underset{\downarrow n=3}{(-1)} e^{-j\frac{2\pi}{5} 3k} + \underset{\downarrow n=4}{1} \cdot e^{-j\frac{2\pi}{5} 4k}$$

$X_p(0) = 1+1-1-1+1 = 1$

$X_p(1) =$

$X_p(2) =$

$X_p(3) =$

$X_p(4) =$

poslije $X_p(4)$ sve se periodično ponavlja

PREDAVANJE

Z TRANSFORMACIJA (ZT)

Definicija:

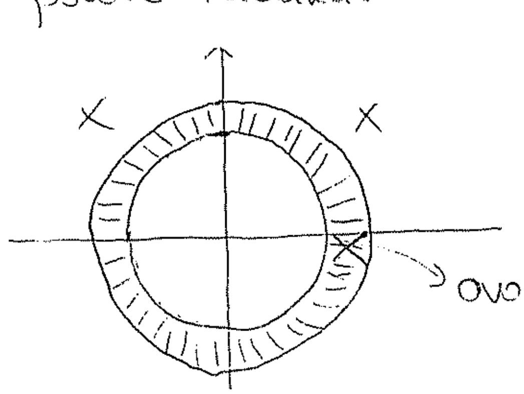
$$\boxed{Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}}$$

Z transformacija postoji u tačkama z i to u onim tačkama u kojima ovaj red konvergira i to se naziva TAČKA KONVERCIJE.

NAPOMENA:

$$X(z) = \frac{P(z)}{Q(z)}$$

Polovi se dobijaju za $Q(z) \equiv 0$, u tim tačkama dobijamo polove tačaka.



ovo je pol

PRIMJER:

Dat je niz $x(n)=u(n)$ naći $X(z)=?$

$$X(z)=\sum_{n=-\infty}^{\infty}u(n)z^{-n}=\underbrace{\sum_{n=0}^{\infty}z^{-n}}_{\text{ovo je geometrijski red}}=\frac{1}{1-z^{-1}}=\frac{1}{1-\frac{1}{z}}=\frac{z}{z-1}$$

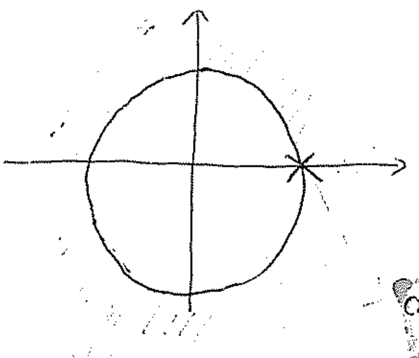
pol $z-1=0$

$z_p=1$

$|z^{-1}|<1$ ili $\left|\frac{1}{z}\right|<1$ ili $|z|>1$

opšti član

ovo je oblast konvergencije a to je izvan kruga

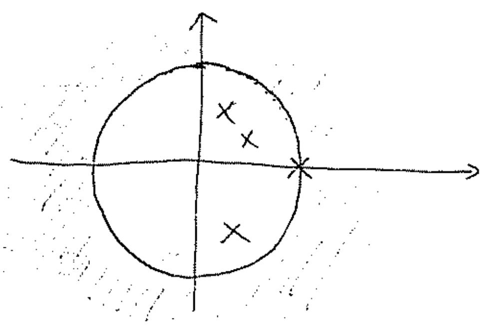


ovo je pol

ovo je oblast konvergencije (izvan kruga)

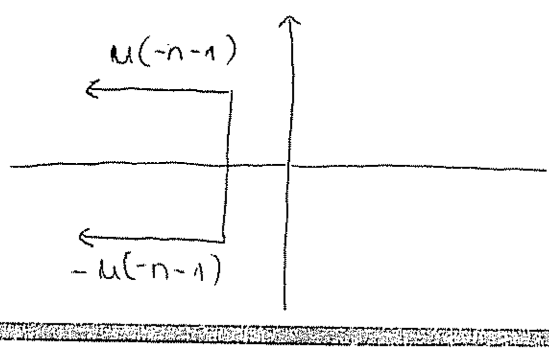
ZAKLJUČAK:

Kod kauzalnih nizova oblast konvergencije je izvan kružnice koja prolazi kroz najudaljeniji pol.



PRIMJER:

Dat je niz $x(n)=-u(-n-1)$ naći $X(z)=?$



$$X(z) = \sum_{n=-\infty}^{\infty} -u(-n-1) z^{-n} = \sum_{n=-\infty}^{-1} -z^{-n} \underset{m=-n}{=} -\sum_{m=1}^{\infty} z^{m} =$$

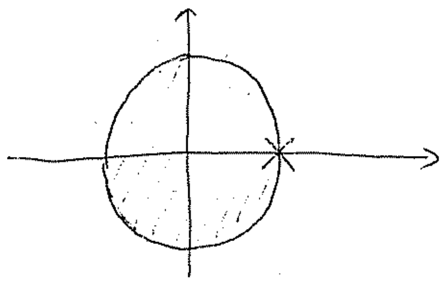
$$= -\sum_{m=0}^{\infty} z^{m+1} = -\sum_{m=0}^{\infty} z^{m} \cdot z = -z \sum_{m=0}^{\infty} z^{m} =$$

$$\underset{\substack{|z|<1 \\ \text{oblast konvergencije}}}{=} -z \frac{1}{1-z} = -\frac{z}{1-z} = \boxed{\frac{z}{z-1}}$$

ovo je z

$z_p = 1$

$$\boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{kada je } |x| < 1}$$

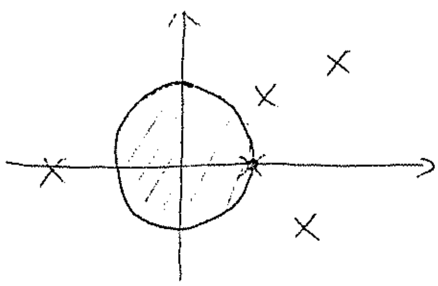


→ oblast konvergencije

$z_p = 1$

ZA ANTIKAUZALNI NIZ oblast konvergencije je unutar kružnice koja prolazi kroz pol koji je najbliži koordinatnom početku.

$\Downarrow$



NAPOMENA:

$Z[u(n)] = \frac{z}{z-1}$, $|z| > 1$ ⇒ oblast konvergencije

$Z[-u(-n-1)] = \frac{z}{z-1}$, $|z| < 1$

Z transformacija jednoznačno opisuje niz kada se uzme njena oblast konvergencije zajedno sa njenim analitičkim os- obrazom.

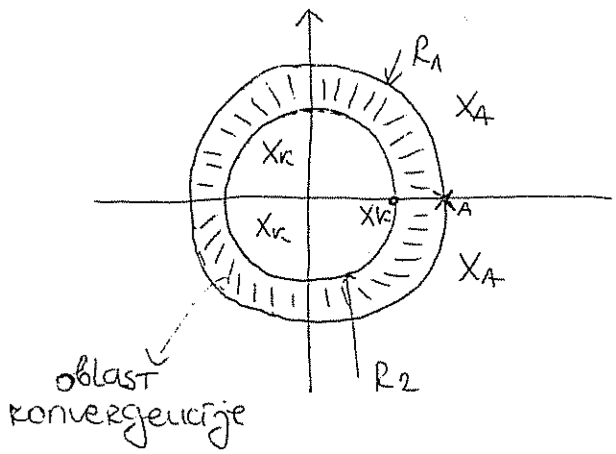
1° KAUZALNI NIZ

2° ANTIKAUZALNI NIZ

3° NEOGRANIČEN SA OBJE STRANE se posmatra kao:

kauzalni + antikauzalni niz

i njegova oblast konvergencije je:



$$R_1 < |z| < R_2$$

X_A - polovi antikauzalnog niza

X_K - polovi kauzalnog niza

<u>Primjer:</u>

$$x(n) = a^n u(n) - b^n u(-n-1)$$

$$X(z) = ?$$

kauzalni dio $\quad x_K(n) = a^n u(n)$

antikauzalni dio $\quad x_A(n) = -b^n u(-n-1)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} =$$

$$= \sum_{n=-\infty}^{-1} -b^n z^{-n} + \sum_{n=0}^{\infty} a^n z^{-n} =$$

$$\underset{-n=m}{=} - \sum_{m=1}^{\infty} b^{-m} z^m + \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n =$$

za $\left|\frac{a}{z}\right| < 1$

$|z| > |a|$

oblast konverg. kauzalnog dijela niza

$$= -\sum_{m=1}^{\infty}\left(\frac{z}{b}\right)^m + \frac{1}{1-\frac{a}{z}} =$$

$$= -\sum_{m=0}^{\infty}\left(\frac{z}{b}\right)^{m+1} + \frac{z}{z-a} =$$

$$= -\frac{z}{b}\sum_{m=0}^{\infty}\left(\frac{z}{b}\right)^{m} + \frac{z}{z-a} =$$

$$= -\frac{z}{b}\cdot\frac{1}{1-\frac{z}{b}} + \frac{z}{z-a} =$$

za $\left|\frac{z}{b}\right|<1$

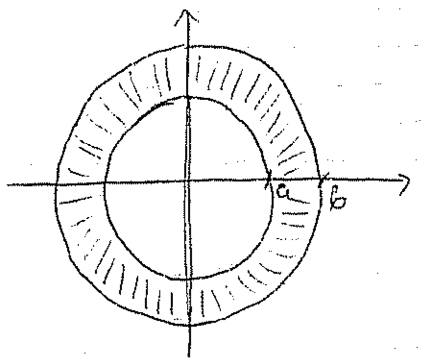
$\boxed{|z|<|b|}$

oblast antikauz. djela niza

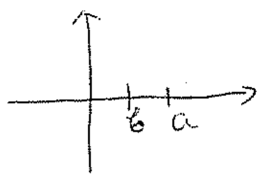
$$= -\frac{z}{b}\cdot\frac{b}{b-z} + \frac{z}{z-a} =$$

$$= \frac{z}{z-b} + \frac{z}{z-a}, \quad \boxed{|a|<|z|<|b|}$$

oblast konvergencije



Da je



oblas konvergencije bi bila izvan i nebi se vidjela.

Stabilnost sistema

opisujemo ga sa $h(n) \xrightarrow[\text{ili}]{} H(z)$

Stabilnost sistema je kada je oblast konvergencije $H(z)$ takva da njoj pripada jedinična kružnica.

PRIMJER:

Data je:

$$H(z) = \frac{z^2 - z + 1}{(z^2 - z + \frac{1}{2})(z - 2)}$$

Odrediti oblast konvergencije tako da sistem bude stabilan.

1° $z^2 - z + \frac{1}{2} = 0$

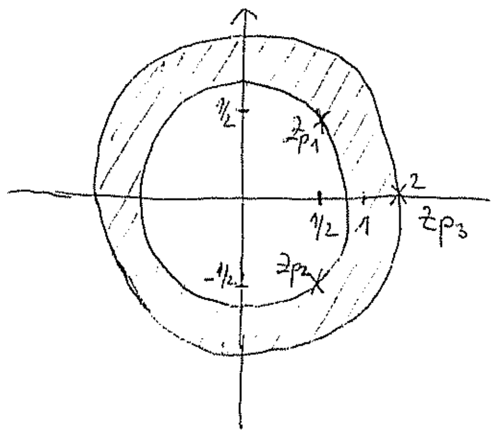
$$z_{p1,2} = \frac{1 \pm \sqrt{1-2}}{2}$$

$$z_{p1,2} = \frac{1 \pm j}{2}$$

$$z_{p1} = \frac{1+j}{2} \qquad z_{p2} = \frac{1-j}{2}$$

2° $z - 2 = 0$

$$z_{p3} = 2$$



$$\boxed{\frac{1}{\sqrt{2}} < |z| < 2}$$

$$|z_{p12}| = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

INVERZNA ZT

$$X(z) = - \quad . = \sum_{n=0}^{\infty} x_n z^{-n}$$

$$\Downarrow$$

$$x(n) = x_n$$

PRIMJER

Odrediti niz $x(n)$ čija je z transformacija:

$$X(z) = \frac{1}{1 - \frac{1}{4} z^{-1}} \qquad x(n) = ?$$

Pol se dobija: $1-\frac{1}{4}z^{-1}=0$

$$\frac{1}{4z}=1$$

$$4z=1$$

$$\boxed{z_p=\frac{1}{4}} \quad \text{ovo je pol}$$

a) $|z|>\frac{1}{4}$ ovo je kauzalni niz

b) $|z|<\frac{1}{4}$ ovo je antikauzalni niz

a) $X(z)=\frac{1}{1-\frac{1}{4}z^{-1}} = \sum_{n=0}^{\infty}\left(\frac{1}{4}z^{-1}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n \cdot z^{-n}$

ovo je Reda

$$\left|\frac{1}{4z}\right|<1$$

$\underbrace{X(n)=\left(\frac{1}{4}\right)^n u(n)}$ ovo je taj niz kada je niz kauzalan

b) $X(z)=\frac{1}{1-\frac{1}{4}z^{-1}}=\frac{1}{1-\frac{1}{4z}}=\frac{4z}{4z-1}=$

$$=-4z\cdot\frac{1}{1-4z}=-4z\sum_{n=0}^{\infty}(4z)^n=$$

$$=-\sum_{n=0}^{\infty}(4z)^{n+1}=-\sum_{n=1}^{\infty}(4z)^n=$$

$m=-n$

$$=-\sum_{m=-\infty}^{-1}(4z)^{-m}=-\sum_{m=-\infty}^{-1}4^{-m}\cdot z^{-m}$$

$X(n)=-4^{-m}\cdot u(-n-1)$

za antikauzalni

Osobine ZT

1° Linearnost

$$y(n) = a x(n) + b h(n)$$

$$\Downarrow$$

$$Y(z) = a X(z) + b H(z)$$

2° Pomjeranje

$$Z[x(n-n_0)] = z^{-n_0} X(z)$$

Primjer:

$$x(n-1) - 2x(n-2) = y(n) + y(n+1)$$

$$\underbrace{X(z) \cdot z^{-1}}_{\text{pomjereno za 1}} - 2X(z) \cdot z^{-2} = Y(z) + Y(z) \cdot z$$

$$X(z)(z^{-1} - 2z^{-2}) = Y(z)(1+z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} - 2z^{-2}}{1+z}$$

3° Množenje eksponencijalnim nizom

$$Z[a^n x(n)] = X\left(\frac{z}{a}\right)$$

4° Konvolucija nizova

$$y(n) = x(n) * h(n)$$

$$\Downarrow$$

$$Y(z) = X(z) \cdot H(z)$$

Primjer:

Odrediti signal na izlaz sistema ako je ulazni signal

$x(n) = u(n)$

$h(n) = \left(\frac{1}{3}\right)^n u(n)$

$$y(n) = x(n) * h(n)$$

$$= \mathcal{Z}^{-1}\left[Y(z) = X(z)H(z)\right]$$

$$X(z) = \frac{z}{z-1}, \quad |z|>1$$

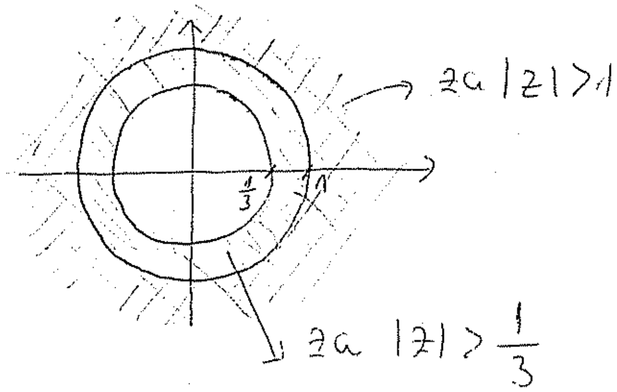
$$H(z) = \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n z^{-n} = \sum_{n=0}^{\infty}\left(\frac{1}{3z}\right)^n = \frac{1}{1-\frac{1}{3z}} = \frac{3z}{3z-1}$$

$$\left|\frac{1}{3z}\right| < 1$$

$$|z| > \frac{1}{3}$$

$$Y(z) = X(z)H(z)$$

$$= \frac{z}{z-1}\cdot\frac{3z}{3z-1} = \frac{3z^2}{(z-1)(3z-1)}, \quad |z|>1$$



$$Y(z) = \frac{3z^2}{(z-1)(3z-1)}, \quad |z|>1 \Rightarrow y(n) = ?$$

$$Y(z) = \frac{Az}{z-1} + \frac{Bz}{3z-1} = \frac{3Az^2 - 1z + 3z^2 - Bz}{(z-1)(3z-1)} =$$

$$= \frac{(3A+B)z^2 - (A+B)z}{(z-1)(3z-1)}$$

$$\left.\begin{array}{l} 3A + B = 3 \\ -A + B = 0 \end{array}\right) -$$

$$2A = 3 \qquad A = \frac{3}{2}$$

$$B = -A$$
$$B = -\frac{3}{2}$$
$$Y(z) = \frac{3}{2} \cdot \frac{z}{z-1} - \frac{3}{2} \cdot \frac{z}{3z-1}$$
$$Y(z) = \frac{3}{2} \cdot \frac{1}{1-\frac{1}{z}} - \frac{3}{2} \cdot \frac{\frac{1}{3}}{1-\frac{1}{3z}}$$
$$= \frac{3}{2} \cdot \sum_{n=0}^{\infty} z^{-n} - \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n z^{-n}$$
$$= \sum_{n=0}^{\infty} \left(\frac{3}{2} - \frac{1}{2}\left(\frac{1}{3}\right)^n\right) z^{-n}$$
$$X(n) = \left(\frac{3}{2} - \frac{1}{2} \cdot \left(\frac{1}{3}\right)^n\right) u(n)$$

$\boxed{\text{VJEŽBE}}$

① Odrediti vremenske nizove kojima je z transformacija data sa:

ⓐ $X(z) = -\frac{2z^2+2.5z}{z^2-4.5z+2}, \quad \frac{1}{2} < |z| < 4$

$$X(z) = -\frac{2z^2+2.5z}{\underbrace{\left(z-\frac{1}{2}\right)(z-4)}_{\text{rješenje jednačine}}}$$

$$X(z) = \frac{Az}{z-\frac{1}{2}} + \frac{Bz}{z-4} = \frac{Az^2 - 4Az + Bz^2 - \frac{1}{2}Bz}{\left(z-\frac{1}{2}\right)(z-4)} =$$

$$= \frac{z^2(A+B) + z\left(-4A - \frac{1}{2}B\right)}{\left(z-\frac{1}{2}\right)(z-4)}$$

$$\left.\begin{aligned} A+B &= -2 \\ -4A - \frac{1}{2}B &= -\frac{5}{2} \end{aligned}\right\} \Rightarrow \begin{aligned} A &= 1 \\ B &= -3 \end{aligned}$$

$$X(z) = \frac{z}{z-\frac{1}{2}} - \frac{3z}{z-4} = \frac{z}{z\left(1-\frac{1}{2z}\right)} + \frac{3z}{4\left(1-\frac{z}{4}\right)} =$$

$$= \underbrace{\frac{1}{1-\frac{1}{2z}}}_{>0} + \underbrace{\frac{\frac{3}{4}z}{1-\frac{z}{4}}}_{>0}$$

kauzalni

$$\left|\frac{1}{2z}\right| < 1 \qquad \left|\frac{z}{4}\right| < 1$$

$$|2\cdot z| > 1 \qquad \boxed{|z| < 4}$$

$$\boxed{|z| > \frac{1}{2}}$$

$$\frac{1\cdot z}{1-\frac{z}{4}} = 3\sum_{n=1}^{\infty}\left(\frac{z}{4}\right)^{n} = 3\sum_{n=-\infty}^{-1}\left(\frac{z}{4}\right)^{-n} \Rightarrow \underbrace{3\cdot 4^{n} u(-n-1)}_{x(n)}$$

$$x(n) = \left(\frac{1}{2}\right)^{n} u(n) + 3\cdot 4^{n} u(-n-1)$$

(6) $$X(z) = \frac{z^2}{(z-1)(z+1)} = \frac{Az}{z-1} + \frac{Bz}{z+1} =$$

$$= \frac{Az^2 + Az + Bz^2 - Bz}{(z-1)(z+1)} =$$

$$= \frac{z^2(A+B) + z(A-B)}{(z-1)(z+1)}$$

$$\left.\begin{aligned} A+B &= 1 \\ A-B &= 0 \end{aligned}\right\} \quad A = B = \frac{1}{2}$$

$$X(z) = \frac{1}{2}\cdot\frac{z}{z-1} + \frac{1}{2}\cdot\frac{z}{z+1}$$

$$\geq 0 \qquad z+1 \geq 0$$

kauzalnost $z_p = 1$ $\qquad z_p = -1$

Za kauzalni sistem $|z|>1$ dobija se Z transformacija:

$$X(z) = \frac{1}{2} \cdot \frac{z}{z(1-\frac{1}{z})} + \frac{1}{2} \cdot \frac{z}{z(1+\frac{1}{z})}$$

$$= \frac{1}{2} \cdot \frac{1}{1-\frac{1}{z}} + \frac{1}{2} \frac{1}{1+\frac{1}{z}}$$

$$x(n) = \frac{1}{2} u(n) + \frac{1}{2}(-1)^n u(n)$$

$$= \frac{1}{2}\left(1+(-1)^n\right) u(n)$$

(2) Diskretni vremenski invarijantni sistem opisan je diferentskom jednačinom:

$$y(n) - \frac{3}{4} y(n-1) + \frac{1}{8} y(n-2) = 2x(n) - \frac{3}{4} x(n-1)$$

Odrediti impulsni odziv sistema koji zadovoljava i uslov da bude i kauzalan.

$$y(n) - \frac{3}{4} y(n-1) + \frac{1}{8} y(n-2) = 2x(n) - \frac{3}{4} x(n-1)$$

$$Y(z) - \frac{3}{4} z^{-1} Y(z) + \frac{1}{8} z^{-2} Y(z) = 2X(z) - \frac{3}{4} X(z) z^{-1}$$

$$Y(z)\left(1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}\right) = X(z)\left(2 - \frac{3}{4} z^{-1}\right)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2 - \frac{3}{4} z^{-1}}{1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}} =$$

$$= \frac{\frac{8z-3}{4z}}{\frac{8z^2-6z+1}{8z^2}} =$$

$8z^2 - 6z + 1 \Rightarrow z_1 = \frac{1}{2}$

$z_2 = \frac{1}{4}$

$$= \frac{16z^2 - 6z}{8(z-\frac{1}{2})(z-\frac{1}{4})}$$

VJEŽBE

① Odrediti impulsni odziv kauzalnog sistema opisanog diferencnom jednačinom:

$$x(n)+2x(n-1)=\frac{1}{4}y(n)-\frac{5}{4}y(n-1)+y(n-2)$$

Da li je sistem stabilan?

$$x(n)+2x(n-1)=\frac{1}{4}y(n)-\frac{5}{4}y(n-1)+y(n-2) \quad \Big| Z_T$$

$$\cancel{X(z)}+2z^{-1}X(z)=\frac{1}{4}Y(z)-\frac{5}{4}z^{-1}Y(z)+z^{-2}Y(z)$$

$$X(z)(1+2z^{-1})=Y(z)\left(\frac{1}{4}-\frac{5}{4}z^{-1}+z^{-2}\right)$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{1+2z^{-1}}{\frac{1}{4}-\frac{5}{4}z^{-1}+z^{-2}} \quad \Rightarrow \text{odavde nalazimo } H(n)$$

$$h(n)=Z_T^{-1}[H(z)]$$

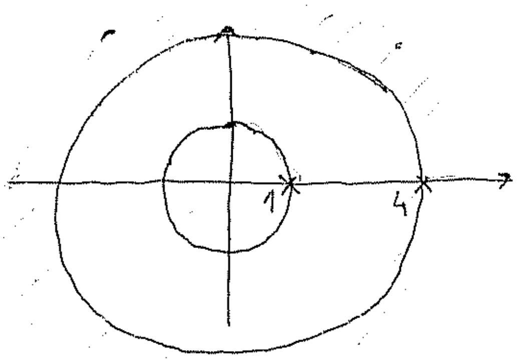
$$H(z)=\frac{z^2+2z}{\frac{1}{4}z^2-\frac{5}{4}z+1}=\frac{4z^2+8z}{z^2-5z+4}$$

Nalazimo polove funkcije $H(z)$

$$z^2-5z+4=0$$

$$z_{p12}=\frac{5\pm\sqrt{25-16}}{2}=\frac{5\pm3}{2}$$

$\boxed{z_{p1}=4}$ $\boxed{z_{p2}=1}$



$$\left.\begin{aligned}|z|&>4\\ |z^{-1}|&<\frac{1}{4}\end{aligned}\right\} \text{oblast konvergencije}$$

Jedinični krug nije iz oblasti konvergencije. Slijedi da sistem nije stabilan.

$$H(z) = \frac{4z^2+8z}{(z-4)(z-1)}$$

$$H(z) = \frac{Az}{z-1} + \frac{Bz}{z-4} = \frac{Az^2-4Az+Bz^2-Bz}{(z-1)(z-4)} =$$

$$= \frac{\overset{A}{(A+B)}z^2+\overset{B}{(-4A-B)}z}{(z-1)(z-4)}$$

treba da svisti

$A+B=4 \Rightarrow$ ono što je uz A

$-4A-B=8 \Rightarrow$ ono što je uz B

$-3A=12$

$$\boxed{A=-4}$$

$B=-4-A$

$B=4-(-4)$

$$\boxed{B=8}$$

$$H(z) = -\frac{4z}{z-1} + \frac{8z}{z-4} \Rightarrow \text{mora da bude veće od } |z|>4$$

kauzalni dio niza

Ako radimo sa kauzalnim sistemom on mora gdje je god prosti razlomak mora biti zapisan kao z^{-1}.

Ako radimo sa antikauzalnim sistemom (djelom niza) on mora gdje je god prosti razlomak mora biti zapisan kao z.

$$H(z) = \frac{4}{1-z^{-1}} + \frac{8}{1-4z^{-1}}$$

$$H(z) = -\frac{4}{1-z^{-1}} + 8\frac{1}{1-4z^{-1}}$$

$$H(z) = -4\cdot\sum_{n=0}^{\infty} z^{-n} + 8\sum_{n=0}^{\infty}(4z^{-1})^n$$

$$H(z) = \sum_{n=0}^{\infty} (-4 + 8 \cdot 4^n)\, z^{-n}$$

$$H(z) \stackrel{def.}{=\!=} \sum h(n)\, z^{-n}$$

$$\boxed{h(n) = (-4 + 8 \cdot 4^n)\, u(n)}$$

(2) Naći ZT kao i pripadajuću oblast konvergencije diskretnog signala:

$$x(n) = 3^n u(-n) + 2^{-n} u(n)$$

$$X(z) = \sum_{-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{-\infty}^{0} 3^n z^{-n} + \sum_{0}^{\infty} 2^{-n} z^{-n}$$

$$= \sum_{-\infty}^{0} \left(\frac{3}{z}\right)^n + \sum_{0}^{\infty} \left(\frac{1}{2z}\right)^n$$

$$\boxed{\sum_{0}^{\infty} q^n = \frac{1}{1-q} \quad \Downarrow \quad |q| < 1}$$

uvodimo smjenu $n = -n$

$$= \sum_{0}^{\infty} \left(\frac{3}{z}\right)^{-n} + \sum_{0}^{\infty} \left(\frac{1}{2z}\right)^n$$

$$= \sum_{0}^{\infty} \left(\frac{z}{3}\right)^n + \sum_{0}^{\infty} \left(\frac{1}{2z}\right)^n$$

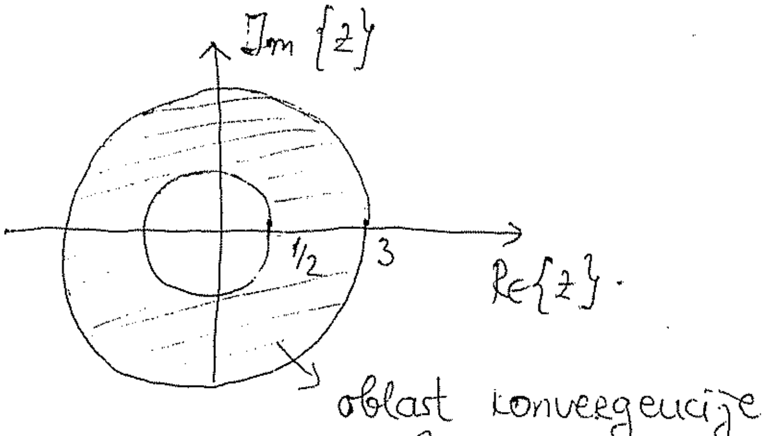
$$= \frac{1}{1 - \frac{z}{3}} + \frac{1}{1 - \frac{1}{2z}}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\left|\frac{z}{3}\right| < 1 \qquad\qquad \left|\frac{1}{2z}\right| < 1$$

$$|z| < 3 \qquad\qquad |2z| > 1$$

$$|z| > \frac{1}{2}$$



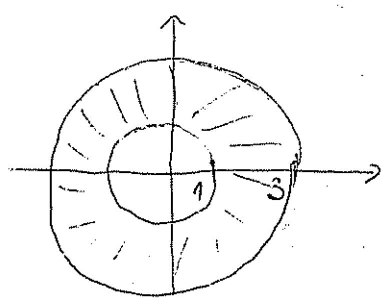
$$\boxed{\frac{1}{2} < |z| < 3}$$

③ Naći inverznu z transformaciju ako je

$$X(z) = \frac{5z-9}{(z-1)(z-3)}$$

ako znamo da je signal $x(n)$ neograničen sa obje strane.

$z_{p1} = 1$
$z_{p2} = 3$



$$|z| > 1 \Rightarrow \left|\frac{1}{z}\right| < 1$$

$$|z| < 3 \Rightarrow \left|\frac{z}{3}\right| < 1$$

$$X(z) = \sum_{-\infty}^{+\infty} x(n) z^{-n}$$

$$X(z) = \frac{5z-9}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3} = \frac{2}{z-1} + \frac{3}{z-3} =$$

$$= \frac{2}{z\left(1-\frac{1}{z}\right)} + \frac{3}{3\left(\frac{z}{3}-1\right)} = \frac{2}{z} \cdot \frac{1}{1-\frac{1}{z}} - \frac{1}{1-\frac{z}{3}} =$$

$$= \frac{2}{z} \cdot \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n = \qquad 2^{-n} \cdot 5^{-n}$$

$$n = -n$$

$$= \frac{2}{z} \sum_{n=0}^{\infty} (1)^n z^{-n} - \sum_{n=-\infty}^{0}\left(\frac{z}{3}\right)^n =$$

$$= 2\sum_{n=0}^{\infty} z^{-(n+1)} - \sum_{n=-\infty}^{0} 3^n z^{-n} =$$

$$= 2\sum_{n=1}^{\infty} z^{-n} - \sum_{n=-\infty}^{0} 3^n z^{-n}$$

$$x(n) = 2 \cdot u(\underbrace{n-1}_{\text{zato jer počinje od } n=1}) - 3^n u(-n)$$

④ Kauzalni sistem je opisan diferencnom jednačinom

$$y(n) = x(n) + \frac{3}{4}x(n-1) - \frac{1}{2}x(n-2) + \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2)$$

Odrediti funkciju prenosa, polove funkcije prenosa i impulsni odziv sistema. Da li je sistem stabilan.

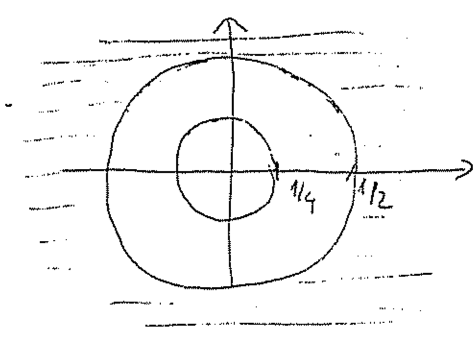
$$Y(z) = X(z) + \frac{3}{4}z^{-1}X(z) - \frac{1}{2}z^{-2}X(z) + \frac{3}{4}z^{-1}Y(z) - \frac{1}{8}z^{-2}Y(z)$$

$$Y(z)\left(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right) = X(z)\left(1 + \frac{3}{4}z^{-1} - \frac{1}{2}z^{-2}\right)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{3}{4}z^{-1} - \frac{1}{2}z^{-2}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \frac{z}{z - \frac{1}{2}} + \frac{1}{z - \frac{1}{4}}$$

$$z_{p1} = \frac{1}{2}$$

$$z_{p2} = \frac{1}{4}$$



$$|z| > \frac{1}{2} \qquad \left|\frac{1}{2z}\right| < 1$$

$$|2z| > 1$$

$$H(z) = \frac{z}{z\left(1-\frac{1}{2z}\right)} + \frac{1}{z\left(1-\frac{1}{4z}\right)}$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2z}\right)^n + \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{4z}\right)^n$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^{-n} + \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n z^{-(n+1)}$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^{n-1} z^{-n}$$

$$h(n) = \left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{4}\right)^{n-1} u(n-1)$$

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